The Drinfeld Realization of the Elliptic Quantum Group $\mathcal{B}_{q,\lambda}(A_2^{(2)})$

Takeo KOJIMA * and Hitoshi KONNO **

* *Department of Mathematics, College of Science and Technology, Nihon University, Chiyoda-ku, Tokyo 101-0062, Japan.*
E-mail:kojima@math.cst.nihon-u.ac.jp

** *Department of Mathematics, Faculty of Integrated Arts and Sciences, Hiroshima University, Higashi-Hiroshima 739-8521, Japan.*
E-mail:konno@mis.hiroshima-u.ac.jp

Abstract

We construct a realization of the L-operator satisfying the RLL-relation of the face type elliptic quantum group $\mathcal{B}_{q,\lambda}(A_2^{(2)})$. The construction is based on the elliptic analogue of the Drinfeld currents of $U_q(A_2^{(2)})$, which forms the elliptic algebra $U_{q,p}(A_2^{(2)})$. We give a realization of the elliptic currents $E(z), F(z)$ and $K(z)$ as a tensor product of the Drinfeld currents of $U_q(A_2^{(2)})$ and a Heisenberg algebra. In the level-one representation, we also give a free field realization of the elliptic currents. Applying these results, we derive a free field realization of the $U_{q,p}(A_2^{(2)})$-analogue of the $\mathcal{B}_{q,\lambda}(A_2^{(2)})$-intertwining operators. The resultant operators coincide with those of the vertex operators in the dilute A_L model, which is known to be a RSOS restriction of the $A_2^{(2)}$ face model.

1 Introduction

An elliptic quantum groups is a quasi-triangular quasi-Hopf algebra obtained as a quasi-Hopf deformation of the affine quantum group $U_q(\mathfrak{g})$ by the twistor satisfying the shifted cocycle condition [1, 2, 3]. It is conjectured in [4, 3] that the representation theory of the elliptic quantum groups of both the vertex type $\mathcal{A}_{q,p}(\widehat{\mathfrak{sl}}_N)$ and the face type $\mathcal{B}_{q,\lambda}(\mathfrak{g})$, $\mathfrak{g}$ being an affine Lie algebra, enables us to perform an algebraic analysis of the corresponding two dimensional solvable lattice models in the sense of Jimbo and Miwa [5]. In order to perform the analysis, we need to construct explicit representations of both finite and infinite dimensional. For this purpose, the Drinfeld realization of the quantum groups is known to provide a relevant framework. In the previous papers[6, 7, 8], we constructed the Drinfeld realization of the face type elliptic quantum group $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_N)$ based on the elliptic algebra $U_{q,p}(\widehat{\mathfrak{sl}}_N)$. The Drinfeld generators have both finite and infinite dimensional representations suitable for the calculation of the correlation functions.

In this paper, we investigate the same problem for $\mathcal{B}_{q,\lambda}(A_2^{(2)})$, the face type elliptic quantum group associated with the twisted affine Lie algebra $A_2^{(2)}$. We first construct the elliptic algebra $U_{q,p}(A_2^{(2)})$ as the algebra of the elliptic analogue of the Drinfeld currents of $U_q(A_2^{(2)})$. Basically, the idea given in Appendix A of [7] can be applied to the twisted case. Namely, dressing the Drinfeld currents of $U_q(A_2^{(2)})$ by the bosons a_m $(m \in \mathbb{Z}_{\neq 0})$ in $U_q(A_2^{(2)})$ and taking a tensor product with a certain Heisenberg algebra $\mathbb{C}\{\mathcal{H}\}$ generated by P, Q, which commutes with $U_q(A_2^{(2)})$, we obtain the elliptic Drinfeld currents. However, we formulate the elliptic algebra $U_{q,p}(A_2^{(2)})$ in an extended form by introducing the new currents $K(u)$, which enables the RLL-formulation of $U_{q,p}(A_2^{(2)})$. Then we discuss a connection between $U_{q,p}(A_2^{(2)})$ and $\mathcal{B}_{q,\lambda}(A_2^{(2)})$. We derive the dynamical RLL-relation of $\mathcal{B}_{q,\lambda}(A_2^{(2)})$ from the RLL-relation of $U_{q,p}(A_2^{(2)})$ by removing a half of the generator Q of the Heisenberg algebra and identifying P with the dynamical parameter in $\mathcal{B}_{q,\lambda}(A_2^{(2)})$. We hence find a structure of $U_{q,p}(A_2^{(2)})$ roughly given by "$\mathcal{B}_{q,\lambda}(A_2^{(2)}) \otimes \mathbb{C}\{\mathcal{H}\}$", and in this sense, we regard $U_{q,p}(A_2^{(2)})$ as the Drinfeld realization of $\mathcal{B}_{q,\lambda}(A_2^{(2)})$.

Although the above tensor structure does not preserve the coalgebra structure of $\mathcal{B}_{q,\lambda}(A_2^{(2)})$, the same tensor structure enables us to convert the algebraic objects of $\mathcal{B}_{q,\lambda}(A_2^{(2)})$, such as the intertwining operators, to the $U_{q,p}(A_2^{(2)})$ counterparts. In the known cases, it is true that the $U_{q,p}(\mathfrak{g})$ counterparts of the $\mathcal{B}_{q,\lambda}(\mathfrak{g})$ intertwining operators play the role

of vertex operators in the restricted solid on solid (RSOS) model associated with $\mathfrak{g}$. We call such "intertwining" operator of $U_{q,p}(\mathfrak{g})$ the vertex operator of $U_{q,p}(\mathfrak{g})$. Moreover, the elliptic Drinfeld currents in $U_{q,p}(A_2^{(2)})$ admits a free field realization, which is an elliptic extension of those of $U_q(A_2^{(2)})$ obtained in [9, 10, 11]. By using such realization and applying the tensoring procedure, we derive a free field realization of the vertex operators of $U_{q,p}(A_2^{(2)})$.

The face model associated with the twisted affine Lie algebra $A_2^{(2)}$ was formulated in [12]. Its RSOS restriction is known to be the dilute A_L model[13, 14]. The free filed formulation of the dilute A_L model was carried out in [15]. There, however, the construction of the vertex operators was done by brute force based on the commutation relations among the vertex operators and on a partial result on the elliptic Drinfeld currents. We here derive the same vertex operators by using the the representation theory of $\mathcal{B}_{q,\lambda}(A_2^{(2)})$ and the Drinfeld realization given by $U_{q,p}(A_2^{(2)})$.

This paper is organized as follows. In the next section, we give a summary of the basic facts on the face type elliptic quantum group $\mathcal{B}_{q,\lambda}(A_2^{(2)})$. In Section 3, we present a definition and a realization of the elliptic algebra $U_{q,p}(A_2^{(2)})$. New currents $K(u)$ are introduced there. In Section 4, we introduce a set of half currents defined from the elliptic currents in $U_{q,p}(A_2^{(2)})$ and derive their commutation relations. Section 5 is devoted to a construction of a L-operator and the RLL-formulation of $U_{q,p}(A_2^{(2)})$. We then derive the dynamical RLL-relation of $\mathcal{B}_{q,\lambda}(A_2^{(2)})$ from $U_{q,p}(A_2^{(2)})$. According to this result, in Section 6, we discuss a free field realization of the two types of vertex operators of the level one $U_{q,p}(A_2^{(2)})$-modules. The final section is devoted to a discussions on some remaining problems. In addition, we have three appendices. In Appendix A, we give a summary of the 3 dimensional evaluation representation of $U_{q,p}(A_2^{(2)})$. In Appendix B, we discuss the difference equation for the twistor and give a partial results on the solutions. Finally, in Appendix C, we give a proof of some formulae of commutation relations of the half currents.

2 The Elliptic Quantum Group $\mathcal{B}_{q,\lambda}(A_2^{(2)})$

In this section, we summarize some basic facts on the face type elliptic quantum group $\mathcal{B}_{q,\lambda}(A_2^{(2)})$ based on the results in [3].

2.1 Notations

Through this article, we fix a complex number $q \neq 0, 0 < q < 1$ and p given by

$$p = q^{2r}, \quad p^* = pq^{-2c} = q^{2r^*} \quad (r^* = r - c;\ r, r^* \in \mathbb{R}_{>0}).$$

We parametrize p as follows.

$$\begin{aligned} &p = e^{-2\pi i/\tau},\ p^* = e^{-2\pi i/\tau^*} \quad (r\tau = r^*\tau^*), \\ &z = q^{2u} = e^{-2\pi i u/r\tau}. \end{aligned}$$

We often use the following Jacobi theta functions.

$$\begin{aligned} [u] &= q^{\frac{u^2}{r}-u}\Theta_p(q^{2u}) = e^{-\frac{\pi i}{4}}\tau^{\frac{1}{2}}q^{-\frac{r}{4}}\ \vartheta_1\left(\frac{u}{r}\middle|\tau\right), \\ [u]_+ &= q^{\frac{u^2}{r}-u}\Theta_p(-q^{2u}) = e^{-\frac{\pi i}{4}}\tau^{\frac{1}{2}}q^{-\frac{r}{4}}\ \vartheta_0\left(\frac{u}{r}\middle|\tau\right), \end{aligned}$$

$[u]^* = [u]|_{r\to r^*,\tau\to\tau^*}$ and $[u]_+^* = [u]_+|_{r\to r^*,\tau\to\tau^*}$. Here

$$\begin{aligned} &\Theta_p(z) = (z,p)_\infty (pz^{-1};p)_\infty (p;p)_\infty, \\ &(z;t_1,\cdots,t_k)_\infty = \prod_{n_1,\cdots,n_k\geq 0}(1 - zt_1^{n_1}\cdots t_k^{n_k}). \end{aligned}$$

The theta functions satisfy $[-u] = -[u], [-u]_+ = [u]_+$ and the quasi-periodicity property

$$[u+r] = -[u], \quad [u+r\tau] = -e^{-\pi i\tau - \frac{2\pi i u}{r}}[u], \tag{2.1}$$

$$[u+r]_+ = [u]_+, \quad [u+r\tau]_+ = e^{-\pi i\tau - \frac{2\pi i u}{r}}[u]_+, \tag{2.2}$$

$$[u + \frac{r\tau}{2}] = ie^{-\pi i(u/r+\tau/4)}[u]_+. \tag{2.3}$$

We use the following normalization for the contour integration.

$$\oint_{C_0}\frac{dz}{2\pi i z}\frac{1}{[-u]} = 1, \qquad \oint_{C_0}\frac{dz}{2\pi i z}\frac{1}{[-u]^*} = \frac{[u]}{[u]^*}\Big|_{u\to 0} \tag{2.4}$$

where C_0 is a simple closed curve in the u-plane encircling $u = 0$ anticlockwise.

2.2 Definition of the elliptic quantum group $\mathcal{B}_{q,\lambda}(A_2^{(2)})$

Let $U_q(A_2^{(2)})$ be the standard affine quantum group, associated with the Cartan matrix

$$A = \begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix}. \tag{2.5}$$

The label of A is $a_0 = 1, a_1 = 2$ and colabel is $a_0^\vee = 2, a_1^\vee = 1$. Let $B = (b_{ij})$ be the symmetrized Cartan matrix $b_{ij} = \frac{a_i^\vee}{a_i} a_{ij}$. We identify $\mathfrak{h} = \mathbb{C}\alpha_0^\vee \oplus \mathbb{C}\alpha_1^\vee \oplus \mathbb{C}d$ and $\mathfrak{h}^* = \mathbb{C}\alpha_0 \oplus \mathbb{C}\alpha_1 \oplus \mathbb{C}\Lambda_0$ via the standard invariant bilinear form (,) given on $\mathfrak{h}$ and $\mathfrak{h}^*$ as follows.

$$\begin{aligned}
&(\alpha_i^\vee, \alpha_j^\vee) = a_{ij}\frac{a_j}{a_j^\vee} \quad (0 \le i, j \le 1),\\
&(\alpha_i^\vee, d) = \delta_{i,0}, \qquad (d, d) = 0,\\
&(\alpha_i, \alpha_j) = \frac{a_i^\vee}{a_i} a_{ij} \quad (0 \le i, j \le 1),\\
&(\alpha_i, \Lambda_0) = \delta_{i,0}, \qquad (\Lambda_0, \Lambda_0) = 0,
\end{aligned}$$

The central element is given by $c = 2\alpha_0^\vee + \alpha_1^\vee$. Let us set $\delta = \alpha_0 + 2\alpha_1$. Then the following relations hold.

$$(\delta, d) = 1, \qquad (\delta, \delta) = 0, \qquad (c, d) = 2, \qquad (c, c) = 0.$$

The identification between $\mathfrak{h}$ and $\mathfrak{h}^*$ is given explicitly by $\alpha_i^\vee = \frac{2\alpha_i}{(\alpha_i, \alpha_i)}$, $c = \delta$ and $d = 2\Lambda_0$. Under this, we use $\{\hat{h}_l\}_{l=1,2,3} = \{d, c, \alpha_1^\vee\}$ as a basis of $\mathfrak{h}$ and its dual basis $\{\hat{h}^l\}_{l=1,2,3} = \{c/2, d/2, \alpha_1^\vee/2\}$. Our conventions of coalgebra structure of $U_q(A_2^{(2)})$ follows [3]. The coproduct, counit, antipode are denoted by Δ, ε and S, respectively.

The face type elliptic quantum group $\mathcal{B}_{q,\lambda}(A_2^{(2)})$ is a quasi-triangular quasi-Hopf algebra obtained from $U_q(A_2^{(2)})$ by the deformation via the face type twistor $F(\lambda)$ ($\lambda \in \mathfrak{h}$). The twistor $F(\lambda)$ is an invertible element in $U_q(A_2^{(2)}) \otimes U_q(A_2^{(2)})$ satisfying

$$(\mathrm{id} \otimes \varepsilon)F(\lambda) = 1 = F(\lambda)(\varepsilon \otimes \mathrm{id}), \tag{2.6}$$

$$F^{(12)}(\lambda)(\Delta \otimes \mathrm{id})F(\lambda) = F^{(23)}(\lambda + h^{(1)})(\mathrm{id} \otimes \Delta)F(\lambda). \tag{2.7}$$

where $\lambda = \sum_l \lambda_l \hat{h}^l$ ($\lambda_l \in \mathbb{C}$), $\lambda + h^{(1)} = \sum_l (\lambda_l + \hat{h}_l^{(1)})\hat{h}^l$ and $\hat{h}_l^{(1)} = \hat{h}_l \otimes 1 \otimes 1$. An explicit construction of the twistor $F(\lambda)$ is given in [3]. As an associative algebra, $\mathcal{B}_{q,\lambda}(A_2^{(2)})$ is

isomorphic to $U_q(A_2^{(2)})$, but the coalgebra structure is deformed in the following way.

$$\Delta_\lambda(x) = F(\lambda)\Delta(x)F(\lambda)^{-1} \qquad \forall x \in U_q(A_2^{(2)}). \tag{2.8}$$

Δ_λ satisfies a weaker coassociativity

$$(\mathrm{id}\otimes\Delta_\lambda)\Delta_\lambda(x) = \Phi(\lambda)(\Delta_\lambda\otimes\mathrm{id})\Delta_\lambda(x)\Phi(\lambda)^{-1} \qquad \forall\ x \in U_q(A_2^{(2)}), \tag{2.9}$$

$$\Phi(\lambda) = F^{(23)}(\lambda)F^{(23)}(\lambda+h^{(1)})^{-1}. \tag{2.10}$$

Let $\mathcal{R}$ be the universal R matrix of $U_q(A_2^{(2)})$. The universal R matrix of $\mathcal{B}_{q,\lambda}(A_2^{(2)})$ is given by

$$\mathcal{R}(\lambda) = F^{(21)}(\lambda)\mathcal{R}F^{(12)}(\lambda)^{-1}. \tag{2.11}$$

Definition 2.1 (Elliptic quantum group $\mathcal{B}_{q,\lambda}(A_2^{(2)})$) *The face type elliptic quantum group $\mathcal{B}_{q,\lambda}(A_2^{(2)})$ is a quasi-triangular quasi-Hopf algebra $(\mathcal{B}_{q,\lambda}(A_2^{(2)}),\ \Delta_\lambda, \varepsilon,\ S,\ \Phi(\lambda),\ \alpha,\ \beta,\ \mathcal{R}(\lambda))$, where $\alpha,\ \beta$ are defined by*

$$\alpha = \sum_i S(k_i)l_i, \quad \beta = \sum_i m_i S(n_i). \tag{2.12}$$

Here we set $\sum_i k_i\otimes l_i = F(\lambda)^{-1},\ \sum_i m_i\otimes n_i = F(\lambda)$.

The universal R matrix $\mathcal{R}(\lambda)$ satisfies the dynamical Yang-Baxter equation.

$$\mathcal{R}^{(12)}(\lambda+h^{(3)})\mathcal{R}^{(13)}(\lambda)\mathcal{R}^{(23)}(\lambda+h^{(1)}) = \mathcal{R}^{(23)}(\lambda)\mathcal{R}^{(13)}(\lambda+h^{(2)})\mathcal{R}^{(12)}(\lambda). \tag{2.13}$$

Let $(\pi_{V,z}, V_z),\ V_z = V\otimes\mathbb{C}[z,z^{-1}]$ be a (finite dimensional) evaluation representation of U_q. Taking images of $\mathcal{R}$, we define a R-matrix $R^+_{VW}(z,\lambda)$ and a L-operator $L^+_V(z,\lambda)$ as follows.

$$R^+_{VW}(z_1/z_2,\lambda) = (\pi_{V,z_1}\otimes\pi_{W,z_2})\, q^{c\otimes d+d\otimes c}\mathcal{R}(\lambda), \tag{2.14}$$

$$L^+_V(z,\lambda) = (\pi_{V,z}\otimes\mathrm{id})\, q^{c\otimes d+d\otimes c}\mathcal{R}(\lambda). \tag{2.15}$$

Then from (2.13), we have the following dynamical RLL-relation.

$$R^+_{VW}(z_1/z_2,\lambda+h)L^+_V(z_1,\lambda)L^+_W(z_2,\lambda+h^{(1)}) = L^+_W(z_2,\lambda)L^+_V(z_1,\lambda+h^{(2)})R^+_{VW}(z_1/z_2,\lambda). \tag{2.16}$$

Note that in $\mathcal{B}_{q,\lambda}(A_2^{(2)})$, $L_V^+(z,\lambda)$ and $L_V^-(z,\lambda) = (\pi_{V,z} \otimes \mathrm{id})\,\mathcal{R}^{(21)}(\lambda)^{-1} q^{-c\otimes d - d\otimes c}$ are not independent operators (Proposition 4.3 in [3]). Hence one dynamical RLL-relation (2.16) characterizes the algebra $\mathcal{B}_{q,\lambda}(A_2^{(2)})$ completely in the sense of Reshetikhin and Semenov-Tian-Shansky [16].

Through this paper, we parametrize the dynamical variable λ as

$$\lambda = (r^* + 3)d + s'c + \frac{1}{2}\left(s + \frac{r\tau}{2}\right)\alpha_1^\vee \qquad (r^* \equiv r - c). \tag{2.17}$$

Under this, we set $F(r^*, s) \equiv F(\lambda)$ and $\mathcal{R}(r^*, s) \equiv \mathcal{R}(\lambda)$. Since c is central, no s' dependence should appear. The dynamical shift $\lambda \to \lambda + h$ with $h = cd + (\alpha_1^\vee)^2/2$, changes the universal R-matrix $\mathcal{R}(r^*, s)$ to $\mathcal{R}(r, s + \alpha_1^\vee)$. Let us take $(\pi_{V,z}, V_z)$ to be the evaluation representation associated with the vector representation $V \cong \mathbb{C}^3$ of $U_q(A_2^{(2)})$ (see Appendix A). We set

$$R^+(u, s + \alpha_1^\vee) = (\pi_{V,z_1} \otimes \pi_{V,z_2}) q^{c\otimes d + d\otimes c} \mathcal{R}(r, s + \alpha_1^\vee),$$
$$L^+(u, s) = (\pi_{V,z} \otimes \mathrm{id}) q^{c\otimes d + d\otimes c} \mathcal{R}(r^*, s),$$

where $z_1/z_2 = q^{2u}$, $u = u_1 - u_2$. From (2.11), we can obtain an explicit expression of $R^+(u,s)$, if we know the finite dimensional representation of the twistor $(\pi_{V,z_1} \otimes \pi_{V,z_2})F(r,s)$. In principle, one can obtain such representation by solving the q-difference equation for the twistor [3], which is similar to the q-KZ equation for corresponding $U_q(\mathfrak{g})$. In the present case, the q-difference equation splits into the three parts; two 2×2 matrix parts and one 3×3 matrix part (see Appendix B). Each 2×2 matrix parts turns out to be the same as the one of the twistor for $\mathcal{B}_{q,\lambda}(A_1^{(1)})$ in the vector representation after adjusting some q-shift and sign factor, whereas we have no known solutions for the 3×3 matrix part. Writing down the solutions of the 2×2 matrix parts under the parametrization of λ (2.17), we obtain from (2.11) the corresponding matrix elements of $R^+(u,s)$ which coincide with the corresponding matrix elements of the Boltzmann weight for the $A_2^{(2)}$ face model[12]. For the remaining 3×3 matrix part, we conjecture that the same coincidence should occur. We hence assume that the R-matrix $R^+(u,s)$ is given by the following formula.

$$R^+(u,s) = \rho^+(u)\bar{R}(u,s), \tag{2.18}$$

where

$$
\bar{R}(u,s) = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & R^{+0}_{+0} & 0 & R^{0+}_{+0} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & R^{+-}_{+-} & 0 & R^{00}_{+-} & 0 & R^{-+}_{+-} & 0 & 0 \\
0 & R^{+0}_{0+} & 0 & R^{0+}_{0+} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & R^{+-}_{00} & 0 & R^{00}_{00} & 0 & R^{-+}_{00} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & R^{0-}_{0-} & 0 & R^{-0}_{0-} & 0 \\
0 & 0 & R^{+-}_{-+} & 0 & R^{00}_{-+} & 0 & R^{-+}_{-+} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & R^{0-}_{-0} & 0 & R^{-0}_{-0} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}, \qquad (2.19)
$$

$$
\begin{aligned}
R^{+0}_{+0}(u,s) &= -\frac{[s+3/2]_+[s-1/2]_+}{[s+1/2]^2_+}\frac{[u]}{[u+1]}, \\
R^{0+}_{+0}(u,s) &= \frac{[s+1/2+u]_+[1]}{[s+1/2]_+[1+u]}, \\
R^{+0}_{0+}(u,s) &= \frac{[-s-1/2+u]_+[1]}{[-s-1/2]_+[1+u]}, \\
R^{0+}_{0+}(u,s) &= R^{-0}_{-0}(u,s) = -\frac{[u]}{[1+u]}, \\
R^{0-}_{0-}(u,s) &= -\frac{[s+1/2]_+[s-3/2]_+}{[s-1/2]^2_+}\frac{[u]}{[u+1]}, \\
R^{-0}_{0-}(u,s) &= \frac{[s-1/2+u]_+[1]}{[s-1/2]_+[1+u]}, \\
R^{0-}_{-0}(u,s) &= \frac{[-s+1/2+u]_+[1]}{[-s+1/2]_+[1+u]}, \\
R^{+-}_{+-}(u,s) &= G^+_s G^-_s \frac{[1/2+u][u]}{[3/2+u][1+u]}, \\
R^{00}_{+-}(u,s) &= -G^-_s \frac{[s+1/2]_+[-s-1-u]_+[1][u]}{[-s+1/2]^2_+[1+u][u+3/2]}, \\
R^{-+}_{+-}(u,s) &= \frac{[-2s+1-u][1]}{[-2s+1][1+u]} - G^-_s \frac{[-2s-1/2-u][u][1]}{[-2s+1][3/2+u][1+u]}, \\
R^{-+}_{00}(u,s) &= -\frac{[-s-1-u]_+[1][u]}{[s+1/2]_+[1+u][u+3/2]}, \\
R^{-+}_{-+}(u,s) &= \frac{[1/2+u][u]}{[3/2+u][1+u]}, \\
R^{00}_{-+}(u,s) &= -\frac{[s-1-u]_+[1][u]}{[-s+1/2]_+[1+u][u+3/2]},
\end{aligned}
$$

$$R^{+-}_{-+}(u,s) = \frac{[2s+1-u][1]}{[2s+1][1+u]} - G^+_s \frac{[2s-1/2-u][u][1]}{[2s+1][3/2+u][1+u]},$$
$$R^{+-}_{00}(u,s) = -G^+_s \frac{[-s+1/2]_+[s-1-u]_+[1][u]}{[s+1/2]^2_+[1+u][u+3/2]},$$
$$R^{00}_{00}(u,s) = \frac{[3+u][1][3/2-u]}{[3][1+u][3/2+u]} + H_s \frac{[1][u]}{[3][1+u]}.$$

Here we have set

$$G^{\pm}_s = -\frac{[2s\pm 2][s]_+}{[2s][s\pm 1]_+}, \quad H_s = G^+_s \frac{[s-5/2]_+}{[s+1/2]_+} + G^-_s \frac{[s+5/2]_+}{[s-1/2]_+}. \tag{2.20}$$

The function $\rho^+(u)$ is given by

$$\rho^+(u) \quad = \quad -qz^{\frac{1}{r}} \frac{\{pq^2z\}\{pq^3z\}\{pq^3z\}\{pq^4z\}\{1/z\}\{q/z\}\{q^5/z\}\{q^6/z\}}{\{pz\}\{pqz\}\{pq^5z\}\{pq^6z\}\{q^2/z\}\{q^3/z\}\{q^3/z\}\{q^4/z\}}, \tag{2.21}$$

where $z = q^{2u}$ and

$$\{z\} = (z; p, q^6)_\infty. \tag{2.22}$$

The R-matrix $R^{+*}(u,s) = (\pi_{V,z_1} \otimes \pi_{V,z_2})\mathcal{R}(r^*,s)$ is obtained from $R^+(u,s)$ by the replacements $r \to r^*$. Hence, under the parametrization (2.17), the dynamical RLL-relation takes the form

$$R^{+(12)}(u, s+\alpha_1^\vee)L^{+(1)}(u_1,s)L^{+(2)}(u_2, s+\alpha_1^{\vee(1)}) = L^{+(2)}(u_2,s)L^{+(1)}(u_1, s+\alpha_1^{\vee(2)})R^{+*(12)}(u,s). \tag{2.23}$$

2.3 Intertwining operators

Let $\mathcal{F}, \mathcal{F}'$ be the highest weight U_q-modules. We denote the type-I and type II intertwining operators of U_q-modules by $\Phi(z)$ and $\Psi^*(z)$, respectively.

$$\Phi(z): \ \mathcal{F} \longrightarrow \mathcal{F}' \otimes W_z, \qquad \Psi^*(z): \ W_z \otimes \mathcal{F} \longrightarrow \mathcal{F}'. \tag{2.24}$$

Twisting these operators by $F(r^*,s)$, we obtain the corresponding intertwining operators $\Phi(v,s)$ and $\Psi^*(u,s)$ of $\mathcal{B}_{q,\lambda}$-modules.

$$\Phi_W(u,s) = (\mathrm{id} \otimes \pi_{W,z})F(r^*,s)\Phi(z), \tag{2.25}$$
$$\Psi^*_W(u,s) = \Psi^*(z)(\pi_{W,z} \otimes \mathrm{id})F(r^*,s)^{-1}. \tag{2.26}$$

From the intertwining relation satisfied by $\Phi(z)$ and $\Psi^*(z)$, one can derive the following dynamical intertwining relation for the new intertwiners [3].

$$\Phi_W^{(3)}(u_2+\frac{c}{2},s)L_V^{+(1)}(u_1,s)=R_{VW}^{+(13)}(u,s+\alpha_1^\vee)L_V^{+(1)}(u_1,s)\Phi_W^{(3)}(u_2+\frac{c}{2},s+\alpha_1^{\vee(1)}), \tag{2.27}$$

$$L_V^{+(1)}(u_1,s)\Psi_W^{*(2)}(z_2,s+\alpha_1^{\vee(1)})=\Psi_W^{*(2)}(z_2,s)L_V^{+(1)}(u_1,s+\alpha_1^{\vee(2)})R_{VW}^{+*(12)}(u_1-u_2,s). \tag{2.28}$$

Note that (2.27) and (2.28) are the relations for the operators $V_{z_1}\otimes\mathcal{F}\to V_{z_1}\otimes\mathcal{F}\otimes W_{z_2}$ and $V_{z_1}\otimes W_{z_2}\otimes\mathcal{F}\to V_{z_1}\otimes\mathcal{F}$, respectively.

3 Elliptic Algebra $U_{q,p}(A_2^{(2)})$

In this section, we give a definition of the elliptic algebra $U_{q,p}(A_2^{(2)})$. We follows the idea given in [7, 8], where the elliptic algebras $U_{q,p}(\mathfrak{g})$ with non-twisted affine Lie algebra $\mathfrak{g}$ are discussed. We first introduce the currents $e(z,p), f(z,p)$ and $\psi^\pm(z,p)$ of the quantum group $U_q(A_2^{(2)})$, by modifying the Drinfeld currents of $U_q(A_2^{(2)})$. We then introduce the new current $k(z)$ in $U_q(A_2^{(2)})$ which is a more basic object than the currents $\psi^\pm(z,p)$. Finally modifying them by taking a tensor product with Heisenberg algebra, we introduce the elliptic currents $E(u), F(u), H^\pm(u)$ and $K(u)$ forming the elliptic algebra $U_{q,p}(A_2^{(2)})$. The current $K(u)$ plays an essential role in the RLL-formulation of $U_{q,p}(A_2^{(2)})$. Hereafter we set $h=\alpha_1^\vee$.

3.1 Drinfeld Currents of $U_q(A_2^{(2)})$

Let us recall the Drinfeld currents of $U_q(A_2^{(2)})$. Let $0<q<1$. We use the standard symbol of q-integer

$$[n]_q=\frac{q^n-q^{-n}}{q-q^{-1}}. \tag{3.1}$$

Definition 3.1 (Drinfeld currents) *Let $x_m^\pm$ $(m\in\mathbb{Z})$, $a_m(m\in\mathbb{Z}_{\neq 0})$ q^c, q^h, q^d denote the Drinfeld generator of $U_q(A_2^{(2)})$. In terms of the generating functions*

$$x^\pm(z)=\sum_{m\in\mathbb{Z}}x_m^\pm z^{-m}, \tag{3.2}$$

$$\psi(q^{c/2}z) = q^{h/2} \exp\left((q-q^{-1})\sum_{m>0} a_m z^{-m}\right), \tag{3.3}$$

$$\varphi(q^{-c/2}z) = q^{-h/2} \exp\left(-(q-q^{-1})\sum_{m>0} a_{-m} z^{m}\right). \tag{3.4}$$

the defining relations of $U_q(A_2^{(2)})$ are given by

$$q^c : \text{central}, \quad q^d a_m q^{-d} = q^m a_m, \quad q^d q_m^{\pm} q^{-d} = q^m x_m^{\pm}, \tag{3.5}$$

$$q^h x^{\pm}(z) q^{-h} = q^{\pm 2} x^{\pm}(z), \quad q^d q^h = q^h q^d. \tag{3.6}$$

$$[a_m, a_n] = \delta_{m+n,0} \frac{1}{m}([2m]_q - [m]_q) q^{-c|m|}[cm]_q, \tag{3.7}$$

$$[a_m, x^+(z)] = \frac{1}{m}([2m]_q - [m]_q) q^{-c|m|} z^m x^+(z), \tag{3.8}$$

$$[a_m, x^-(z)] = -\frac{1}{m}([2m]_q - [m]_q) z^m x^-(z), \tag{3.9}$$

$$(z_1 - q^{\pm 2} z_2)(z_1 - q^{\mp 1} z_2) x^{\pm}(z_1) x^{\pm}(z_2) = -(q^{\pm 2} z_1 - z_2)(q^{\mp 1} z_1 - z_2) x^{\pm}(z_2) x^{\pm}(z_1), \tag{3.10}$$

$$[x^+(z_1), x^-(z_2)] = \frac{1}{q-q^{-1}} \left(\psi(q^{c/2} z_2)\delta(q^{-c} z_1/z_2) - \varphi(q^{-c/2} z_2)\delta(q^c z_1/z_2)\right), \tag{3.11}$$

$$\sum_{\sigma \in S_3} \left(q^{\pm 3/2} z_{\sigma(1)} - (q^{1/2} + q^{-1/2}) z_{\sigma(2)} + q^{\pm 3/2} z_{\sigma(3)}\right) x^{\pm}(z_{\sigma(1)}) x^{\pm}(z_{\sigma(2)}) x^{\pm}(z_{\sigma(3)}) = 0. \tag{3.12}$$

Here $\delta(z)$ denotes the delta function $\delta(z) = \sum_{m \in \mathbb{Z}} z^m$. We call the generators $h, a_m, x_m^{\pm}, c, d$ the Drinfeld generators of $U_q(A_2^{(2)})$ and the generating functions $x^{\pm}(z), \psi(z)$ and $\varphi(z)$ the Drinfeld currents.

3.2 Elliptic currents of $U_q(A_2^{(2)})$

We next consider an elliptic modification of the Drinfeld currents $x^{\pm}(z), \psi(z)$ and $\varphi(z)$. Let us introduce the two auxiliary currents $u^{\pm}(z,p)$ by

$$u^+(z,p) = \exp\left(\sum_{m>0} \frac{a_{-m}}{[r^* m]_q} q^{rm} z^m\right), \tag{3.13}$$

$$u^-(z,p) = \exp\left(-\sum_{m>0} \frac{a_m}{[rm]_q} q^{rm} z^{-m}\right). \tag{3.14}$$

Proposition 3.1 *The following commutation relations hold.*

$$u^+(z_1,p)u^-(z_2,p)$$
$$= u^-(z_2,p)u^+(z_1,p)\frac{(pq^{-c-2}z_1/z_2;p)_\infty(p^*q^{c+2}z_1/z_2;p^*)_\infty(pq^{-c+1}z_1/z_2;p)_\infty(p^*q^{c-1}z_1/z_2;p^*)_\infty}{(pq^{-c+2}z_1/z_2;p)_\infty(p^*q^{c-2}z_1/z_2;p^*)_\infty(pq^{-c-1}z_1/z_2;p)_\infty(p^*q^{c+1}z_1/z_2;p^*)_\infty}, \tag{3.15}$$

$$u^+(z_1,p)x^+(z_2) = \frac{(p^*q^2z_1/z_2;p^*)_\infty(p^*q^{-1}z_1/z_2;p^*)_\infty}{(p^*q^{-2}z_1/z_2;p^*)_\infty(p^*qz_1/z_2;p^*)_\infty}x^+(z_2)u^+(z_1,p), \tag{3.16}$$

$$u^+(z_1,p)x^-(z_2) = \frac{(p^*q^{c-2}z_1/z_2;p^*)_\infty(p^*q^{c+1}z_1/z_2;p^*)_\infty}{(p^*q^{c+2}z_1/z_2;p^*)_\infty(p^*q^{c-1}z_1/z_2;p^*)_\infty}x^-(z_2)u^+(z_1,p), \tag{3.17}$$

$$u^-(z_1,p)x^+(z_2) = \frac{(pq^{-c-2}z_2/z_1;p)_\infty(pq^{-c+1}z_2/z_1;p)_\infty}{(pq^{-c+2}z_2/z_1;p)_\infty(pq^{-c-1}z_2/z_1;p)_\infty}x^+(z_2)u^-(z_1,p), \tag{3.18}$$

$$u^-(z_1,p)x^-(z_2) = \frac{(pq^2z_2/z_1;p)_\infty(pq^{-1}z_2/z_1;p)_\infty}{(pq^{-2}z_2/z_1;p)_\infty(pqz_2/z_1;p)_\infty}x^-(z_2)u^-(z_1,p), \tag{3.19}$$

$$\psi(z_1,p)u^+(z_2,p)$$
$$= u^+(z_2,p)\psi(z_1,p)\frac{(q^{r^*+2}z_2/z_1;p)_\infty(q^{r^*-1}z_2/z_1;p)_\infty(q^{r^*-2}z_2/z_1;p^*)_\infty(q^{r^*+1}z_2/z_1;p^*)_\infty}{(q^{r^*-2}z_2/z_1;p)_\infty(q^{r^*+1}z_2/z_1;p)_\infty(q^{r^*+2}z_2/z_1;p^*)_\infty(q^{r^*-1}z_2/z_1;p^*)_\infty}, \tag{3.20}$$

$$\psi(z_1,p)u^-(z_2,p)$$
$$= u^-(z_2,p)\psi(z_1,p)\frac{(q^{r-2}z_1/z_2;p)_\infty(q^{r+1}z_1/z_2;p)_\infty(q^{r+2}z_1/z_2;p^*)_\infty(q^{r-1}z_1/z_2;p^*)_\infty}{(q^{r+2}z_1/z_2;p)_\infty(q^{r-1}z_1/z_2;p)_\infty(q^{r-2}z_1/z_2;p^*)_\infty(q^{r+1}z_1/z_2;p^*)_\infty},$$

$$\psi(z_1,p)x^+(z_2)$$
$$= x^+(z_2)\psi(z_1,p)\frac{(q^{r^*-2}z_2/z_1;p)_\infty(q^{r^*+1}z_2/z_1;p)_\infty(q^{r^*+2}z_1/z_2;p^*)_\infty(q^{r^*-1}z_1/z_2;p^*)_\infty}{(q^{r^*+2}z_2/z_1;p)_\infty(q^{r^*-1}z_2/z_1;p)_\infty(q^{r^*-2}z_1/z_2;p^*)_\infty(q^{r^*+1}z_1/z_2;p^*)_\infty}, \tag{3.21}$$

$$\psi(z_1,p)x^-(z_2)$$
$$= x^-(z_2)\psi(z_1,p)\frac{(q^{r+2}z_2/z_1;p)_\infty(q^{r-1}z_2/z_1;p)_\infty(q^{r-2}z_1/z_2;p^*)_\infty(q^{r+1}z_1/z_2;p^*)_\infty}{(q^{r-2}z_2/z_1;p)_\infty(q^{r+1}z_2/z_1;p)_\infty(q^{r+2}z_1/z_2;p^*)_\infty(q^{r-1}z_1/z_2;p^*)_\infty}. \tag{3.22}$$

Definition 3.2 *We define "dressed" currents $e(z,p), f(z,p)$ and $\psi^\pm(z,p)$ by*

$$e(z,p) = u^+(z,p)x^+(z), \tag{3.23}$$

$$f(z,p) = x^-(z)u^-(z,p), \tag{3.24}$$

$$\psi^+(z,p) = u^+(q^{c/2}z,p)\psi(z)u^-(q^{-c/2}z,p), \tag{3.25}$$

$$\psi^-(z,p) = u^+(q^{-c/2}z,p)\varphi(z)u^-(q^{c/2}z,p). \tag{3.26}$$

If we introduce the auxiliary current $\psi(z,p)$ by

$$\psi(z,p) = \exp\left(\sum_{m>0}\frac{x^{cm}}{[r^*m]_q}a_{-m}z^m\right)\exp\left(-\sum_{m>0}\frac{1}{[rm]_q}a_m z^{-m}\right), \tag{3.27}$$

we have

$$\psi^\pm(q^{\mp(r-c/2)}z) = q^{\pm h/2}\psi(z,p). \tag{3.28}$$

Proposition 3.2 *The currents $e(z,p), f(z,p)$ and $\psi(z,p)$ satisfy the following commutation relations.*

$$\psi(z_1,p)\psi(z_2,p) = \frac{\Theta_p(q^{-2}z_1/z_2)\Theta_p(qz_1/z_2)}{\Theta_p(q^2z_1/z_2)\Theta_p(q^{-1}z_1/z_2)}\frac{\Theta_{p^*}(q^2z_1/z_2)\Theta_{p^*}(q^{-1}z_1/z_2)}{\Theta_{p^*}(q^{-2}z_1/z_2)\Theta_{p^*}(qz_1/z_2)}\psi(z_2,p)\psi(z_1,p), \tag{3.29}$$

$$\psi(z_1,p)e(z_2,p) = \frac{\Theta_{p^*}(q^{r^*+2}z_1/z_2)\Theta_{p^*}(q^{r^*-1}z_1/z_2)}{\Theta_{p^*}(q^{r^*-2}z_1/z_2)\Theta_{p^*}(q^{r^*+1}z_1/z_2)}e(z_2,p)\psi(z_1,p), \tag{3.30}$$

$$\psi(z_1,p)f(z_2,p) = \frac{\Theta_p(q^{r-2}z_1/z_2)\Theta_p(q^{r+1}z_1/z_2)}{\Theta_p(q^{r+2}z_1/z_2)\Theta_p(q^{r-1}z_1/z_2)}f(z_2,p)\psi(z_1,p), \tag{3.31}$$

$$e(z_1,p)e(z_2,p) = (-1)\frac{\Theta_{p^*}(q^{-2}z_2/z_1)\Theta_{p^*}(q^{-1}z_1/z_2)}{\Theta_{p^*}(q^{-2}z_1/z_2)\Theta_{p^*}(q^{-1}z_2/z_1)}e(z_2,p)e(z_1,p), \tag{3.32}$$

$$f(z_1,p)f(z_2,p) = (-1)\frac{\Theta_p(q^2z_2/z_1)\Theta_p(qz_1/z_2)}{\Theta_p(q^2z_1/z_2)\Theta_p(qz_2/z_1)}f(z_2,p)f(z_1,p), \tag{3.33}$$

$$[e(z_1,p),f(z_2,p)] = \frac{1}{q-q^{-1}}\left(\psi^+(q^{c/2}z_2)\delta(q^{-c}z_1/z_2) - \psi^-(q^{-c/2}z_2)\delta(q^cz_1/z_2)\right), \tag{3.34}$$

$$\begin{aligned}\sum_{\sigma\in S_3} & \frac{(p^*q^2z_{\sigma(3)}/z_{\sigma(1)};p^*)_\infty(p^*q^{-1}z_{\sigma(3)}/z_{\sigma(1)};p^*)_\infty(p^*q^{-1}z_{\sigma(3)}/z_{\sigma(2)};p^*)_\infty(p^*q^{-1}z_{\sigma(2)}/z_{\sigma(1)};p^*)_\infty}{(p^*q^{-2}z_{\sigma(3)}/z_{\sigma(1)};p^*)_\infty(p^*qz_{\sigma(3)}/z_{\sigma(1)};p^*)_\infty(p^*qz_{\sigma(3)}/z_{\sigma(2)};p^*)_\infty(p^*qz_{\sigma(2)}/z_{\sigma(1)};p^*)_\infty}\\ &\times\left(z_{\sigma(1)}\frac{(q^2z_{\sigma(2)}/z_{\sigma(1)};p^*)_\infty(p^*q^2z_{\sigma(3)}/z_{\sigma(2)};p^*)_\infty}{(p^*q^{-2}z_{\sigma(2)}/z_{\sigma(1)};p^*)_\infty(p^*q^{-2}z_{\sigma(3)}/z_{\sigma(2)};p^*)_\infty}\right.\\ &\left.-qz_{\sigma(2)}\frac{(p^*q^2z_{\sigma(2)}/z_{\sigma(1)};p^*)_\infty(p^*q^2z_{\sigma(3)}/z_{\sigma(2)};p^*)_\infty}{(p^*q^{-2}z_{\sigma(2)}/z_{\sigma(1)};p^*)_\infty(p^*q^{-2}z_{\sigma(3)}/z_{\sigma(2)};p^*)_\infty}\right)e(z_{\sigma(1)},p)e(z_{\sigma(2)},p)e(z_{\sigma(3)},p)=0,\end{aligned} \tag{3.35}$$

$$\begin{aligned}\sum_{\sigma\in S_3} & \frac{(pqz_{\sigma(2)}/z_{\sigma(1)};p)_\infty(pq^{-2}z_{\sigma(3)}/z_{\sigma(1)};p)_\infty(pqz_{\sigma(3)}/z_{\sigma(2)};p)_\infty(pqz_{\sigma(3)}/z_{\sigma(2)};p)_\infty}{(pq^{-1}z_{\sigma(2)}/z_{\sigma(1)};p)_\infty(pq^2z_{\sigma(3)}/z_{\sigma(1)};p)_\infty(pq^{-1}z_{\sigma(3)}/z_{\sigma(1)};p)_\infty(pq^{-1}z_{\sigma(3)}/z_{\sigma(2)};p)_\infty}\\ &\times\left(z_{\sigma(1)}\frac{(q^{-2}z_{\sigma(2)}/z_{\sigma(1)};p)_\infty(pq^{-2}z_{\sigma(3)}/z_{\sigma(2)};p)_\infty}{(pq^2z_{\sigma(2)}/z_{\sigma(1)};p)_\infty(pq^2z_{\sigma(3)}/z_{\sigma(2)};p)_\infty}\right.\\ &\left.-q^{-1}z_{\sigma(2)}\frac{(pq^{-2}z_{\sigma(2)}/z_{\sigma(1)};p)_\infty(q^{-2}z_{\sigma(3)}/z_{\sigma(2)};p)_\infty}{(pq^2z_{\sigma(2)}/z_{\sigma(1)};p)_\infty(pq^2z_{\sigma(3)}/z_{\sigma(2)};p)_\infty}\right)f(z_{\sigma(1)},p)f(z_{\sigma(2)},p)f(z_{\sigma(3)},p)=0.\end{aligned} \tag{3.36}$$

3.3 Basic current $k(z)$

The current $\psi(z,p)$ (3.27) can be expressed by more basic current $k(z)$. Let us introduce new generator of the bosons, α_m,β_m.

$$\alpha_m = \begin{cases} a_m, & m>0,\\ \frac{[rm]_q}{[r^*m]_q}q^{c|m|}a_m, & m<0,\end{cases}\qquad \beta_m = \alpha_m\frac{[r^*m]_q}{[rm]_q}. \tag{3.37}$$

They satisfy the following commutation relations.

$$[\alpha_m,\alpha_n]=\delta_{m+n,0}\frac{[2m]_q-[m]_q}{m}\frac{[cm]_q[rm]_q}{[r^*m]_q}, \tag{3.38}$$

$$[\beta_m,\beta_n]=\delta_{m+n,0}\frac{[2m]_q-[m]_q}{m}\frac{[cm]_q[r^*m]_q}{[rm]_q}, \tag{3.39}$$

The current $\psi(z,p)$ is expressed by

$$\psi(z,p)=:\exp\left(-\sum_{m\neq 0}\frac{\alpha_m}{[rm]_q}z^{-m}\right):=:\exp\left(-\sum_{m\neq 0}\frac{\beta_m}{[r^*m]_q}z^{-m}\right):. \tag{3.40}$$

The colons : : denote the standard normal ordering.

Definition 3.3 (Basic Current) *We define the current $k(z)$ by*

$$k(z,p)=:\exp\left(-\sum_{m\neq 0}\frac{[m]_q}{[rm]_q([2m]_q-[m]_q)}\alpha_m z^{-m}\right):. \tag{3.41}$$

The current $\psi(z,p)$ is expressed by $k(z)$ as follows.

$$\psi(z,p)=:k(q^{-1}z,p)k(z,p)^{-1}k(qz,p):. \tag{3.42}$$

By a straightforward calculation, we have the following commutation relations.

Proposition 3.3

$$k(z_1,p)u^+(z_2,p)=\frac{(q^{r^*+1}z_2/z_1;p)_\infty(q^{r^*-1}z_2/z_1;p^*)_\infty}{(q^{r^*-1}z_2/z_1;p)_\infty(q^{r^*+1}z_2/z_1;p^*)_\infty}u^+(z_2,p)k(z_1,p), \tag{3.43}$$

$$k(z_1,p)u^-(z_2,p)=\frac{(q^{r-1}z_1/z_2;p)_\infty(q^{r+1}z_1/z_2;p^*)_\infty}{(q^{r+1}z_1/z_2;p)_\infty(q^{r-1}z_1/z_2;p^*)_\infty}u^-(z_2,p)k(z_1,p), \tag{3.44}$$

$$k(z_1,p)x^+(z_2)=\frac{(q^{r^*+1}z_1/z_2;p^*)_\infty(q^{r^*-1}z_2/z_1;p)_\infty}{(q^{r^*-1}z_1/z_2;p^*)_\infty(q^{r^*+1}z_2/z_1;p)_\infty}x^+(z_2)k(z_1,p), \tag{3.45}$$

$$k(z_1,p)x^-(z_2)=\frac{(q^{r-1}z_1/z_2;p^*)_\infty(q^{r+1}z_2/z_1;p)_\infty}{(q^{r+1}z_1/z_2;p^*)_\infty(q^{r-1}z_2/z_1;p)_\infty}x^-(z_2)k(z_1,p). \tag{3.46}$$

Proposition 3.4 *The currents $e(z,p), f(z,p)$ and $k(z,p)$ satisfy the following commutation relations.*

$$k(z_1,p)k(z_2,p) = z^{-1/r^*+1/r}\rho(z_1/z_2)k(z_2,p)k(z_1,p), \tag{3.47}$$

$$k(z_1,p)e(z_2,p) = \frac{\Theta_{p^*}(q^{r^*+1}z_1/z_2)}{\Theta_{p^*}(q^{r^*-1}z_1/z_2)}e(z_2,p)k(z_1), \tag{3.48}$$

$$k(z_1,p)f(z_2,p) = \frac{\Theta_p(q^{r-1}z_1/z_2)}{\Theta_p(q^{r+1}z_1/z_2)}f(z_2,p)k(z_1,p). \tag{3.49}$$

Here we have set

$$\rho(z) \;=\; \frac{\rho^{+*}(z)}{\rho^{+}(z)}, \tag{3.50}$$

where $\rho^+(z)$ *is given in (2.21) and* $\rho^{+*}(z) = \rho^+(z)|_{r\to r^*}$.

3.4 Elliptic Algebra $U_{q,p}(A_2^{(2)})$

Now we give a definition of the elliptic algebra $U_{q,p}(A_2^{(2)})$. For this purpose, we introduce a Heisenberg algebra $\mathbb{C}\{\mathcal{H}\}$ generated by P, Q, and $\bar{\alpha}$.

$$[P,Q]=1, \quad [Q,\bar{\alpha}]=\pi i, \quad [P,\bar{\alpha}]=0, \tag{3.51}$$

$$[P,P]=[Q,Q]=[\bar{\alpha},\bar{\alpha}]=0. \tag{3.52}$$

Definition 3.4 (Elliptic Currents) *We define the elliptic (total) currents* $E(z), F(z)$ *and* $K(z)$ *by*

$$E(z) \;=\; e(z)e^{\bar{\alpha}}e^{-Q}z^{-P/r^*}, \tag{3.53}$$

$$F(z) \;=\; f(z)e^{-\bar{\alpha}}z^{P/r+h/2r}, \tag{3.54}$$

$$K(z) \;=\; k(z)e^{-Q}z^{(1/r-1/r^*)P+h/2r}. \tag{3.55}$$

Let us introduce the auxiliary currents $H^{\pm}(z)$ by

$$H^{\pm}(z) = H(q^{\pm(r-c/2)}z), \tag{3.56}$$

$$H(z) = \psi(z)e^{-Q}z^{(1/r-1/r^*)P+h/2r} = \kappa K(qz)K(z)^{-1}K(q^{-1}z), \tag{3.57}$$

where

$$\kappa = \frac{\{pq^8\}\{pq^5\}\{pq^3\}\{pq^4\}^2\{p\}\{p^*q^7\}^*\{p^*q\}^*\{p^*q^2\}^{*2}\{p^*q^6\}^{*2}}{\{pq^7\}\{pq\}\{pq^2\}^2\{pq^6\}^2\{p^*\}^*\{p^*q^8\}^*\{p^*q^5\}^*\{p^*q^3\}^*\{p^*q^4\}^{*2}}. \tag{3.58}$$

From the commutation relations of the currents $e(z,p), f(z,p)$ and $k(z,p)$, we can verify the following relations.

Theorem 3.5 *The elliptic currents* $E(z), F(z)$ *and* $K(z)$ *satisfy the following commutation relations.*

$$K(z_1)K(z_2) \;=\; \rho(z_1/z_2)K(z_2)K(z_1), \tag{3.59}$$

$$K(z_1)E(z_2) = -\frac{[u_1-u_2+\frac{r^*+1}{2}]^*}{[u_1-u_2+\frac{r^*-1}{2}]^*}E(z_2)K(z_1), \tag{3.60}$$

$$K(z_1)F(z_2) = -\frac{[u_1-u_2+\frac{r-1}{2}]}{[u_1-u_2+\frac{r+1}{2}]}F(z_2)K(z_1), \tag{3.61}$$

$$E(z_1)E(z_2) = -\frac{[u_1-u_2+1]^*[u_1-u_2-\frac{1}{2}]^*}{[u_1-u_2-1]^*[u_1-u_2+\frac{1}{2}]^*}E(z_2)E(z_1), \tag{3.62}$$

$$F(z_1)F(z_2) = -\frac{[u_1-u_2-1][u_1-u_2+\frac{1}{2}]}{[u_1-u_2+1][u_1-u_2-\frac{1}{2}]}F(z_2)F(z_1), \tag{3.63}$$

$$[E(z_1),F(z_2)] = \frac{1}{q-q^{-1}}\left(H^+(q^{c/2}z_2)\delta(q^{-c}z_1/z_2)-H^-(q^{-c/2}z_2)\delta(q^{c}z_1/z_2)\right) \tag{3.64}$$

Here $\rho(z)$ is given in (3.50). The elliptic currents $E(z)$ and $F(z)$ satisfy the following Serre relations.

$$\begin{aligned}\sum_{\sigma\in S_3} &\frac{(p^*q^2z_{\sigma(3)}/z_{\sigma(1)};p^*)_\infty(p^*q^{-1}z_{\sigma(3)}/z_{\sigma(1)};p^*)_\infty(p^*q^{-1}z_{\sigma(3)}/z_{\sigma(2)};p^*)_\infty(p^*q^{-1}z_{\sigma(2)}/z_{\sigma(1)};p^*)_\infty}{(p^*q^{-2}z_{\sigma(3)}/z_{\sigma(1)};p^*)_\infty(p^*qz_{\sigma(3)}/z_{\sigma(1)};p^*)_\infty(p^*qz_{\sigma(3)}/z_{\sigma(2)};p^*)_\infty(p^*qz_{\sigma(2)}/z_{\sigma(1)};p^*)_\infty}\\
&\times z_{\sigma(1)}^{-\frac{1}{2r^*}}z_{\sigma(2)}^{-\frac{1}{r^*}}\left(z_{\sigma(1)}\frac{(q^2z_{\sigma(2)}/z_{\sigma(1)};p^*)_\infty(p^*q^2z_{\sigma(3)}/z_{\sigma(2)};p^*)_\infty}{(p^*q^{-2}z_{\sigma(2)}/z_{\sigma(1)};p^*)_\infty(p^*q^{-2}z_{\sigma(3)}/z_{\sigma(2)};p^*)_\infty}\right.\\
&\left.-qz_{\sigma(2)}\frac{(p^*q^2z_{\sigma(2)}/z_{\sigma(1)};p^*)_\infty(p^*q^2z_{\sigma(3)}/z_{\sigma(2)};p^*)_\infty}{(p^*q^{-2}z_{\sigma(2)}/z_{\sigma(1)};p^*)_\infty(p^*q^{-2}z_{\sigma(3)}/z_{\sigma(2)};p^*)_\infty}\right)E(z_{\sigma(1)})E(z_{\sigma(2)})E(z_{\sigma(3)})=0,\end{aligned} \tag{3.65}$$

and

$$\begin{aligned}\sum_{\sigma\in S_3} &\frac{(pqz_{\sigma(2)}/z_{\sigma(1)};p)_\infty(pq^{-2}z_{\sigma(3)}/z_{\sigma(1)};p)_\infty(pqz_{\sigma(3)}/z_{\sigma(2)};p)_\infty(pqz_{\sigma(3)}/z_{\sigma(2)};p)_\infty}{(pq^{-1}z_{\sigma(2)}/z_{\sigma(1)};p)_\infty(pq^2z_{\sigma(3)}/z_{\sigma(1)};p)_\infty(pq^{-1}z_{\sigma(3)}/z_{\sigma(1)};p)_\infty(pq^{-1}z_{\sigma(3)}/z_{\sigma(2)};p)_\infty}\\
&\times z_{\sigma(1)}^{2/r}z_{\sigma(2)}^{1/r}\left(z_{\sigma(1)}\frac{(q^{-2}z_{\sigma(2)}/z_{\sigma(1)};p)_\infty(pq^{-2}z_{\sigma(3)}/z_{\sigma(2)};p)_\infty}{(pq^2z_{\sigma(2)}/z_{\sigma(1)};p)_\infty(pq^2z_{\sigma(3)}/z_{\sigma(2)};p)_\infty}\right.\\
&\left.-q^{-1}z_{\sigma(2)}\frac{(pq^{-2}z_{\sigma(2)}/z_{\sigma(1)};p)_\infty(q^{-2}z_{\sigma(3)}/z_{\sigma(2)};p)_\infty}{(pq^2z_{\sigma(2)}/z_{\sigma(1)};p)_\infty(pq^2z_{\sigma(3)}/z_{\sigma(2)};p)_\infty}\right)F(z_{\sigma(1)})F(z_{\sigma(2)})F(z_{\sigma(3)})=0.\end{aligned} \tag{3.66}$$

Definition 3.5 **(Elliptic Algebra $U_{q,p}(A_2^{(2)})$)** *We define the elliptic algebra $U_{q,p}(A_2^{(2)})$ to be the associative algebra generated by the currents $E(z), F(z)$ and $K(z)$ satisfying the relations (3.56)-(3.66).*

Corollary 3.6 *The construction of $E(z), F(z)$ and $K(z)$ given in (3.53)-(3.55) is a realization of the elliptic algebra $U_{q,p}(A_2^{(2)})$ in terms of the Drinfeld generator of the quantum group $U_q(A_2^{(2)})$ and the Heisenberg algebra $\mathbb{C}\{\mathcal{H}\}$.*

For later convenience, let us introduce auxiliary currents $K_\epsilon(z), (\epsilon = 0, \pm)$ by

$$
\begin{aligned}
K_+(z) &= K(q^{r-2}z) = k(q^{r-2}z)e^{-Q}(q^{r-2}z)^{(1/r-1/r^*)P+h/2r}, && (3.67)\\
K_0(z) &= K(q^r z)^{-1}K(q^{r-1}z) = k(q^r z)^{-1}k(q^{r-1}z)q^{(1/r^*-1/r)P-h/2r}, && (3.68)\\
K_-(z) &= K(q^{r+1}z)^{-1} = k(q^{r+1}z)^{-1}(q^{r+1}z)^{(1/r^*-1/r)P-h/2r}e^Q. && (3.69)
\end{aligned}
$$

Then one can verify the following relations.

Proposition 3.7

$$K_+(z_1)E(z_2) = -\frac{[u_1-u_2+\frac{c-1}{2}]^*}{[u_1-u_2+\frac{c-3}{2}]^*}E(z_2)K_+(z_1), \tag{3.70}$$

$$K_0(z_1)E(z_2) = \frac{[u_1-u_2+\frac{c}{2}]^*[u_1-u_2+\frac{c-1}{2}]^*}{[u_1-u_2+\frac{c}{2}-1]^*[u_1-u_2+\frac{c+1}{2}]^*}E(z_2)K_0(z_1), \tag{3.71}$$

$$K_-(z_1)E(z_2) = -\frac{[u_1-u_2+\frac{c}{2}]^*}{[u_1-u_2+\frac{c}{2}+1]^*}E(z_2)K_-(z_1), \tag{3.72}$$

$$K_+(z_1)F(z_2) = -\frac{[u_1-u_2-\frac{3}{2}]}{[u_1-u_2-\frac{1}{2}]}F(z_2)K_+(z_1), \tag{3.73}$$

$$K_0(z_1)F(z_2) = \frac{[u_1-u_2-1][u_1-u_2+\frac{1}{2}]}{[u_1-u_2][u_1-u_2-\frac{1}{2}]}F(z_2)K_0(z_1), \tag{3.74}$$

$$K_-(z_1)F(z_2) = -\frac{[u_1-u_2+1]}{[u_1-u_2]}F(z_2)K_-(z_1), \tag{3.75}$$

$$H^\pm(q^{\pm c/2}z) = H(q^{\pm r}z) = \kappa K_-(z)^{-1}K_0(z) = \kappa' K_+(qz)K_0(qz)^{-1}. \tag{3.76}$$

Here μ is given in (3.58) and κ' is given by

$$\kappa' = \frac{\{pq^{10}\}\{pq^7\}\{pq^5\}\{pq^6\}^2\{pq^2\}\{p^*q^9\}^*\{p^*q^3\}^*\{p^*q^5\}^{*2}\{p^*q^8\}^{*2}}{\{pq^9\}\{pq^3\}\{pq^5\}^2\{pq^8\}^2\{p^*q^2\}^*\{p^*q^{10}\}^*\{p^*q^7\}^*\{p^*q^5\}^*\{p^*q^6\}^{*2}}. \tag{3.77}$$

4 Half Currents

As a preparation for the RLL-formulation of the elliptic algebra $U_{p,q}(A_2^{(2)})$ in the next section, we here introduce the half currents, and investigate their commutation relations.

Let us first summarize the commutation relations between the Heisenberg algebra $\mathbb{C}\{\mathcal{H}\}$ and the elliptic currents. From (3.53)-(3.55), we have the following relations.

Proposition 4.1

$$[E(z), P] = E(z), \quad [F(z), P + \frac{h}{2}] = F(z), \tag{4.1}$$

$$[E(z), P+\frac{h}{2}]=0,\ \ [F(z),P]=0, \tag{4.2}$$

$$[K_+(z),P]=K_+(z)=[K_+(z),P+\frac{h}{2}], \tag{4.3}$$

$$[K_0(z),P]=0=[K_0(z),P+\frac{h}{2}], \tag{4.4}$$

$$[K_-(z),P]=-K_-(z)=[K_-(z),P+\frac{h}{2}]. \tag{4.5}$$

Now we define the half currents $E^+_{-,0}(u), E^+_{0,+}(u), E^+_{-,+}(u), F^+_{0,-}(u), F^+_{+,0}(u), F^+_{+,-}(u)$ and $K^+_\epsilon(u), (\epsilon=0,\pm)$, by the following formulae.

Definition 4.1 (Half Currents)

$$\begin{aligned}
K^+_\epsilon(u) &= K_\epsilon(z), \quad (\epsilon=0,\pm), && (4.6)\\
E^+_{-,0}(u) &= a^*_{-0}\oint_{C^*_{-0}}\frac{dz'}{2\pi i z'}E(z')\frac{[u-u'-P+\frac{c+1}{2}]^*_+[1]^*}{[u-u'+\frac{c}{2}]^*[P-\frac{1}{2}]^*_+}, && (4.7)\\
E^+_{0,+}(u) &= a^*_{0+}\oint_{C^*_{0+}}\frac{dz'}{2\pi i z'}E(z')\frac{[u-u'-P+\frac{c}{2}]^*_+[1]^*}{[u-u'+\frac{c-1}{2}]^*[P-\frac{1}{2}]^*_+}, && (4.8)\\
E^+_{-,+}(u) &= a^*_{-+}\oint\oint_{C^*_{-+}}\frac{dz'}{z'}\frac{dz''}{z''}E(z')E(z'')\frac{[1]^{*2}}{[P-\frac{1}{2}]^*_+[2P-2]^*}\\
&\times \frac{[u-u'-2P+2+\frac{c}{2}]^*[u'-u''-P]^*_+}{[u-u'+\frac{c}{2}]^*[u'-u''-\frac{1}{2}]^*}, && (4.9)
\end{aligned}$$

$$\begin{aligned}
F^+_{0,-}(u) &= a_{0-}\oint_{C_{0-}}\frac{dz'}{2\pi i z'}F(z')\frac{[u-u'+P+\frac{h-1}{2}]_+[1]}{[u-u'][P+\frac{h-1}{2}]_+}, && (4.10)\\
F^+_{+,0}(z) &= a_{+0}\oint_{C_{+0}}\frac{dz'}{2\pi i z'}F(z')\frac{[u-u'+P+\frac{h}{2}-1]_+[1]}{[u-u'-\frac{1}{2}][P+\frac{h-1}{2}]_+}, && (4.11)\\
F^+_{+,-}(u) &= a_{+-}\oint\oint_{C_{+-}}\frac{dz'}{2\pi i z'}\frac{dz''}{2\pi i z''}F(z')F(z'')\frac{[P+\frac{h}{2}-1]_+[1]^2}{[P+\frac{h-3}{2}]_+[P+\frac{h}{2}-2]_+[2P+h-2]}\\
&\times \frac{[u-u'+2P+h-3][u-u''+1][u'-u''+P+\frac{h}{2}-1]_+}{[u-u'][u-u''][u'-u''+\frac{1}{2}]}. && (4.12)
\end{aligned}$$

Here C^*_{-0} is a simple closed contour that encircles pq^cz but not q^cz. We abbreviate it as $C^*_{-0}:|p^*q^cz|<|z'|<|q^cz|$. Similarly the others are given by

$$\begin{aligned}
C^*_{-0} &: |p^*q^cz|<|z'|<|q^cz|, && (4.13)\\
C^*_{0+} &: |p^*q^{c-1}z|<|z'|<|q^{c-1}z|, && (4.14)\\
C^*_{-+} &: |p^*q^c|<|z'|<|q^cz|,\ |p^*q^cz|<|z''|<|q^cz|,\ |p^*qz'|<|z''|<|qz'|, && (4.15)\\
C_{0-} &: |pz|<|z'|<|z|, && (4.16)
\end{aligned}$$

$$C^*_{0+} \quad : \quad |pq^{-1}z| < |z'| < |q^{-1}z|, \tag{4.17}$$

$$C_{+-} \quad : \quad |pz| < |z'| < |z|, \ |pz| < |z''| < |z|, \ |pqz'| < |z''| < |qz'|. \tag{4.18}$$

The constants $a^*_{-0}, a^*_{0+}, a^*_{-+}, a_{0-}, a_{+0}, a_{+-}$ *are chosen to satisfy*

$$\frac{\mu a_{0-} a^*_{-0}[1]^*}{q-q^{-1}} = -1 = \frac{\mu' a_{+0} a^*_{0+}[1]^*}{q-q^{-1}}, \quad a_{+-} = a_{0-}a_{0-}, \quad a^*_{-+} = a^*_{-0}a^*_{-0}. \tag{4.19}$$

We can verify the following commutation relations.

Theorem 4.2 *The half currents satisfy the following relations.*

$$K^+_{\pm}(u_1)K^+_{\pm}(u_2) = \rho(u)K^+_{\pm}(u_2)K^+_{\pm}(u_1), \tag{4.20}$$

$$K^+_0(u_1)K^+_0(u_2) = \frac{\rho(u)\rho(u)}{\rho(u+\frac{1}{2})\rho(u-\frac{1}{2})}K^+_0(u_2)K^+_0(u_1), \tag{4.21}$$

$$K^+_-(u_1)K^+_+(u_2) = K^+_+(u_2)K^+_-(u_1)\rho(u)\frac{[u_1-u_2+1][u_1-u_2+\frac{3}{2}][u_1-u_2]^*[u_1-u_2+\frac{1}{2}]^*}{[u_1-u_2][u_1-u_2+\frac{1}{2}][u_1-u_2+1]^*[u_1-u_2+\frac{3}{2}]^*}, \tag{4.22}$$

$$K^+_-(u_1)K^+_0(u_2) = \rho(u)\frac{[u_1-u_2]^*[u_1-u_2+1]}{[u_1-u_2+1]^*[u_1-u_2]}K^+_0(u_2)K^+_-(u_1), \tag{4.23}$$

$$K^+_0(u_1)K^+_+(u_2) = \rho(u)\frac{[u_1-u_2]^*[u_1-u_2+1]}{[u_1-u_2+1]^*[u_1-u_2]}K^+_+(u_2)K^+_0(u_1), \tag{4.24}$$

$$K^+_-(u_1)^{-1}E^+_{-,0}(u_2)K^+_-(u_1) = -E^+_{-,0}(u_2)\frac{[u+1]^*}{[u]^*} + E^+_{-,0}(u_1)\frac{[P+\frac{1}{2}+u]^*_+[1]^*}{[P+1/2]^*_+[u]^*}, \tag{4.25}$$

$$K^+_+(u_2)^{-1}E^+_{0,+}(u_1)K^+_+(u_2) = -\frac{[u+1]^*}{[u]^*}E^+_{0,+}(u_1) + \frac{[-P+\frac{1}{2}+u]^*_+[1]^*}{[-P+\frac{1}{2}]^*_+[u]^*}E^+_{0,+}(u_2), \tag{4.26}$$

$$\begin{aligned}
&K^+_-(u_1)^{-1}E^+_{-,+}(u_2)K^+_-(u_1)\\
&\quad = E^+_{-,+}(u_2)\frac{[u+\frac{3}{2}]^*[u+1]^*}{[u+\frac{1}{2}]^*[u]^*} + K^+_-(u_1)^{-1}E^+_{-,0}(u_2)K^+_-(u_1)E^+_{-,0}(u_1)\frac{[-P-1-u]^*_+[1]^*}{[P+\frac{1}{2}]^*_+[u+\frac{1}{2}]^*}\\
&\qquad -E_{-,+}(u_1)\left(\frac{[-2P+1-u]^*[u+\frac{3}{2}]^*[1]^*}{[-2P+1]^*[u+\frac{1}{2}]^*[u]^*} + \frac{[2P-2]^*[P]^*_+[-2P-\frac{1}{2}-u]^*[1]^*}{[2P]^*[P-1]^*_+[-2P+1]^*[u+\frac{1}{2}]^*}\right),
\end{aligned} \tag{4.27}$$

$$K^+_-(u_1)F^+_{0,-}(u_2)K_-(u_1)^{-1} = -\frac{[u+1]}{[u]}F^+_{0,-}(u_2) + \frac{[-P+\frac{-h+1}{2}+u]_+[1]}{[-P+\frac{-h+1}{2}]_+[u]}F^+_{0,-}(u_2), \tag{4.28}$$

$$K^+_+(u_2)F^+_{+,0}(u_1)K^+_+(u_2)^{-1} = -F^+_{+,0}(u_1)\frac{[u+1]}{[u]} + F^+_{+,0}(u_2)\frac{[P+\frac{h+1}{2}+u]_+[1]}{[P+\frac{h+1}{2}]_+[u]}, \tag{4.29}$$

$$\begin{aligned}
&K^+_-(u_1)F^+_{+,-}(u_2)K^+_-(u_1)^{-1}\\
&\quad = \frac{[u+\frac{3}{2}][u+1]}{[u+\frac{1}{2}][u]}F^+_{+,-}(u_2) - \frac{[P+\frac{h}{2}-1-u]_+[1]}{[-P+\frac{-h+1}{2}][u+\frac{1}{2}]}F^+_{0,-}(u_1)K^+_-(u_1)F^+_{0,-}(u_2)K^+_-(u_1)^{-1}\\
&\qquad -\left(\frac{[u+\frac{3}{2}][2P+h+1-u][1]}{[u+\frac{1}{2}][u][2P+h+1]} + \frac{[2P+h-\frac{1}{2}-u][1][2P+h+2][P+\frac{h}{2}]_+}{[2P+h+1][2P+h][P+\frac{h}{2}+1]_+[u+\frac{1}{2}]}\right)F^+_{+,-}(u_1),
\end{aligned}$$

$$(4.30)$$

$$[E^+_{-,0}(u_1), F^+_{0,-}(u_2)]$$
$$= -K^+_0(u_2)\frac{[-P-\frac{1}{2}+u]^*_+[1]^*}{[-P-\frac{1}{2}]^*_+[u]^*}K^+_-(u_2)^{-1} + K^+_-(u_1)^{-1}\frac{[-P+\frac{-h+1}{2}+u]_+[1]}{[-P+\frac{-h+1}{2}]_+[u]}K^+_0(u_1), \quad (4.31)$$

$$[E^+_{0,+}(u_1), F^+_{+,0}(u_2)]$$
$$= -K^+_0(u_2)^{-1}\frac{[P+\frac{h+1}{2}+u]_+[1]}{[P+\frac{h+1}{2}]_+[u+1]}K^+_+(u_2) + K^+_+(u_1)\frac{[P-\frac{1}{2}+u]^*_+[1]^*}{[P-\frac{1}{2}]^*_+[u+1]^*}K^+_0(u_1)^{-1}. \quad (4.32)$$

where $u = u_1 - u_2$.

Proof. The relations (4.20)-(4.24) are direct consequences of the commutation relation of the elliptic current $K(z)$. Let us consider the relations (4.25)-(4.32). These relations can be proved by reducing them to identities of the theta functions. We show the relation (4.25). The relations (4.26), (4.28) and (4.29) can be proved in the same way. From the definition of the half current (4.7) and the commutation relation of (3.72), the LHS of (4.25) yields

$$K^+_-(u_1)E^+_{-,0}(u_2)K^+_-(u_1)^{-1}$$
$$= -a^*_{-,0}\oint_{C^*_{-0}}\frac{dz'}{2\pi i z'}E(z')\frac{[u_1-u'+\frac{c}{2}+1]^*[u_2-u'-P+\frac{c-1}{2}]^*_+[1]^*}{[u_1-u'+\frac{c}{2}]^*[u_2-u'+\frac{c}{2}]^*[P+\frac{1}{2}]^*_+}. \quad (4.33)$$

Then the equality is verified by the following identity of the theta functions.

$$- \frac{[u_1-u'+\frac{c}{2}+1]^*[u_2-u'-P+\frac{c-1}{2}]^*_+}{[u_1-u'+\frac{c}{2}]^*[u_2-u'+\frac{c}{2}]^*[P+\frac{1}{2}]^*_+}$$
$$= -\frac{[u_2-u'-P+\frac{c+1}{2}]^*_+[u_1-u_2+1]^*}{[u_2-u'+\frac{c}{2}]^*[u_1-u_2]^*[P-\frac{1}{2}]^*_+} + \frac{[u_1-u'-P+\frac{c+1}{2}]^*_+[u_1-u_2+P+\frac{1}{2}]^*_+[1]^*}{[u_1-u'+\frac{c}{2}]^*[u_1-u_2]^*[P-\frac{1}{2}]^*_+[P+\frac{1}{2}]^*_+}.$$

Next, we show the relation (4.31). The relation (4.32) can be proved in the same way. Integrating the delta function appearing in (3.64) and using (3.76), we have

$$(\mu a_{0-}a^*_{-0})^{-1}(q-q^{-1})[E^+_{-,0}(u_1), F^+_{0,-}(u_2)]$$
$$= \oint_{C^+}\frac{dz'}{2\pi i z'}K^+_-(u')^{-1}K^+_0(u')\frac{[u_1-u'-P+\frac{1}{2}]^*_+[1]^*[u_2-u'+P+\frac{h-1}{2}]_+[1]}{[u_1-u']^*[P-\frac{1}{2}]^*_+[u_2-u'][P+\frac{h-1}{2}]_+} \quad (4.34)$$
$$- \oint_{C^-}\frac{dz'}{2\pi i z'}K^+_-(u'-r)^{-1}K^+_0(u'-r)\frac{[u_1-u'-P+c+\frac{1}{2}]^*_+[1]^*[u_2-u'+P+\frac{h-1}{2}]_+[1]}{[u_1-u'+c]^*[P-\frac{1}{2}]^*_+[u_2-u'][P+\frac{h-1}{2}]_+}$$

Here the contours $C^\pm$ are now

$$C^+ : \ |p^*z_1|, |pz_2| < |z'| < |z_1|, |z_2|, \quad (4.35)$$

$$C^- : \ |pz_1|, |pz_2| < |z'| < |q^{2c}z_1|, |z_2|. \quad (4.36)$$

When we change the integration variable $z' \to pz'$ in the second term, the integrand becomes the same as the first term, but the contour C^- is changed to $\tilde{C}^-$ given by

$$\tilde{C^-}: \ |z_1|, |z_2| < |z'| < |p^{-1}q^{2c}z_1|, |p^{-1}z_2|. \tag{4.37}$$

Taking the residues at $z' = z_1, z_2$, we get (4.31).

We give a proof of (4.27) in Appendix C. One can prove (4.30) in a similar way.

Q.E.D.

5 The L-operator of $U_{q,p}(A_2^{(2)})$ and Relation to $\mathcal{B}_{q,\lambda}(A_2^{(2)})$

In this section, we clarify the relation between two elliptic algebras $U_{q,p}(A_2^{(2)})$ and $\mathcal{B}_{q,\lambda}(A_2^{(2)})$. For this purpose, we first construct a L-operator which gives the RLL-formulation of $U_{q,p}(A_2^{(2)})$. Then modifying L-operator by removing the Heisenberg generators $Q, \bar{\alpha}$, we derive the dynamical RLL-relation (5.8) characterizing the elliptic quantum group $\mathcal{B}_{q,\lambda}(A_2^{(2)})$.

5.1 L-operator of $U_{q,p}(A_2^{(2)})$

Definition 5.1 *By using the half currents, we define the L-operator $\widehat{L}^+(u) \in \mathrm{End}(\mathbb{C}^3) \otimes U_{q,p}(A_2^{(2)})$ as follows.*

$$\widehat{L}^+(u) = \begin{pmatrix} 1 & F^+_{+0}(u) & F^+_{+-}(u) \\ 0 & 1 & F^+_{0-}(u) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} K^+_{+}(u) & 0 & 0 \\ 0 & K^+_{0}(u) & 0 \\ 0 & 0 & K^+_{-}(u) \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ E^+_{0+}(u) & 1 & 0 \\ E^+_{-+}(u) & E^+_{-0}(u) & 1 \end{pmatrix}. \tag{5.1}$$

Here matrix elements are the half currents given in the previous section.

By a direct comparison with the relations of the half currents appeared in Theorem 4.2, we get the following commutation relations of the L-operator.

Theorem 5.1 *The L-operator $\widehat{L}^+(u)$ satisfies the following $RLL = LLR^*$ relation.*

$$\begin{aligned} &R^{+(12)}(u_1 - u_2, P + h/2)\widehat{L}^{+(1)}(u_1)\widehat{L}^{+(2)}(u_2) \\ &\qquad = \widehat{L}^{+(2)}(u_2)\widehat{L}^{+(1)}(u_1)R^{+*(12)}(u_1 - u_2, P - (h^{(1)} + h^{(2)})/2). \end{aligned} \tag{5.2}$$

The above equation should be understood as equation of the operators acting on the space $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes U_{q,p}(A_2^{(2)})$. The operator h in LHS acts on $U_{q,p}(A_2^{(2)})$, whereas the operator $h^{(1)} + h^{(2)}$ in RHS acts on $\mathbb{C}^3 \otimes \mathbb{C}^3$ as $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix} \otimes 1 + 1 \otimes \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}$.

5.2 $U_{q,p}(A_2^{(2)})$ and $\mathcal{B}_{q,\lambda}(A_2^{(2)})$

Based on the above theorem, we give a relation between $U_{q,p}(A_2^{(2)})$ and $\mathcal{B}_{q,\lambda}(A_2^{(2)})$. We argue that the RLL relation (5.2) is equivalent to the dynamical RLL relation of $\mathcal{B}_{q,\lambda}(A_2^{(2)})$. Hence we can regard the elliptic currents in $U_{q,p}(A_2^{(2)})$ as an elliptic analogue of the Drinfeld currents in $U_q(A_2^{(2)})$ providing a new realization of the elliptic quantum group $\mathcal{B}_{q,\lambda}(A_2^{(2)})$. In order to show this, we consider the realization of $U_{q,p}(A_2^{(2)})$ given in (3.53)-(3.55) and modify the half currents in such a way that they have no $Q, \bar{\alpha}$ dependence.

$$k_+(u,P) = K_+(u)e^{Q}, \quad k_0(u,P) = K_0(u) \quad k_-(u,P) = K_-(u)e^{-Q}, \tag{5.3}$$

$$f_{+,0}(u,P) = e^{\bar{\alpha}}F_{+,-}(u), \quad f_{0,-}(u,P) = e^{\bar{\alpha}}F_{0,-}(u), \quad f_{+,-}(u,P) = e^{\bar{\alpha}}F_{+,-}(u)e^{\bar{\alpha}}, \tag{5.4}$$

$$e_{0,+}(u,P) = E_{0,+}(u)e^{Q}e^{-\bar{\alpha}}, \quad e_{-,0}(u,P) = e^{Q}e^{-\bar{\alpha}}E_{-,0}(u), \quad e_{-,+}(u,P) = e^{Q}e^{-\bar{\alpha}}E_{-,+}(u)e^{Q}e^{-\bar{\alpha}}. \tag{5.5}$$

We regard them as the currents in $U_q(A_2^{(2)})$ with parameters P and r. Then we define a dynamical L-operator $\widehat{L}^+(u,P)$ by

$$\begin{aligned} \widehat{L}^+(u,P) &= \begin{pmatrix} 1 & f^+_{+0}(u,P) & f^+_{+-}(u,P) \\ 0 & 1 & f^+_{0-}(u,P) \\ 0 & 0 & 1 \end{pmatrix} \\ &\times \begin{pmatrix} k^+_+(u,P) & 0 & 0 \\ 0 & k^+_0(u,P) & 0 \\ 0 & 0 & k^+_-(u,P) \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ e^+_{0+}(u,P) & 1 & 0 \\ e^+_{-+}(u,P) & e^+_{-0}(u,P) & 1 \end{pmatrix}. \end{aligned} \tag{5.6}$$

The two L-operators $\widehat{L}^+(u)$ and $\widehat{L}^+(u,P)$ are related by

$$\widehat{L}^+(u,P) = \widehat{L}^+(u) \begin{pmatrix} e^{Q} & & \\ & 1 & \\ & & e^{-Q} \end{pmatrix} = \widehat{L}^+(u)e^{Qh^{(1)}/2}. \tag{5.7}$$

Here $h^{(1)} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix} \otimes 1$. Substituting this into (5.2) and moving the factor $e^{-Qh^{(j)}/2}$ $(j=1,2)$ to the right end in the both sides, we get the following statement.

Corollary 5.2 *The dynamical L-operator $L^+(u,P)$ satisfies the dynamical RLL relation.*

$$\begin{aligned} &R^{+(12)}(u_1-u_2,P+h/2)L^{+(1)}(u_1,P)L^{+(2)}(u_2,P+h^{(1)}/2) \\ &\qquad = L^{+(2)}(u_2,P)L^{+(1)}(u_1,P+h^{(2)}/2)R^{+*(12)}(u_1-u_2,P). \end{aligned} \tag{5.8}$$

Comparing this with (2.23), we identify our $L^+(u,P)$ with $L^+(u,s)$ in (2.23) and s with P. We hence regard the elliptic currents $E(z), F(z)$ and $K(z)$ in $U_{q,p}(A_2^{(2)})$ as the Drinfeld currents of the elliptic quantum group $\mathcal{B}_{q,\lambda}(A_2^{(2)})$, although $U_{q,p}(A_2^{(2)})$ and $\mathcal{B}_{q,\lambda}(A_2^{(2)})$ are different by tensoring the Heisenberg algebra $\mathbb{C}\{\mathcal{H}\}$. More precisely, $U_{q,p}(A_2^{(2)})$ is an extension of the algebra $\mathcal{B}_{q,\lambda}(A_2^{(2)})$ by tensoring the Heisenberg algebra $\mathbb{C}\{\mathcal{H}\}$; first tensoring the generators $e^Q, e^{\bar{\alpha}}$, then regarding $s=P$ and imposing the commutation relations (3.51) and (3.52). Naively we regards $U_{q,p}(A_2^{(2)})$ as $\mathcal{B}_{q,\lambda}(A_2^{(2)}) \otimes_{\mathbb{C}\{P\}} \mathbb{C}\{\mathcal{H}\}$.

6 Vertex Operators

Tensoring the Heisenberg algebra breaks down the coalgebra structure of $\mathcal{B}_{q,\lambda}(A_2^{(2)})$. But we can define the $U_{q,p}(A_2^{(2)})$ counterparts of the intertwining operators of $\mathcal{B}_{q,\lambda}(A_2^{(2)})$. We call such operators the vertex operators of $U_{q,p}(A_2^{(2)})$. In this section, we study such vertex operators and compare them with those of the dilute A_L model obtained in [15].

6.1 Intertwining Relations

We here derive the $U_{q,p}(A_2^{(2)})$ counterparts of the dynamical intertwining relations (2.27)-(2.28). In the next subsection, we use such relations to derive a free field realization of the vertex operators.

Let us first define an extension of the U_q modules by

$$\widehat{\mathcal{F}} = \bigoplus_{\mu\in\mathbb{Z}} \mathcal{F}\otimes e^{\mu Q}.$$

Let $\Phi_W(u,P)$ and $\Psi^*_W(u,P)$ be the type I and type II intertwining operators of $\mathcal{B}_{q,\lambda}(A_2^{(2)})$ (2.27)-(2.28). We define type I and type II vertex operators $\widehat{\Phi}_W(u)$, $\widehat{\Psi}^*_W(u)$ of $U_{q,p}(A_2^{(2)})$ as the following extensions of the corresponding intertwining operators of $\mathcal{B}_{q,\lambda}(A_2^{(2)})$.

$$\widehat{\Phi}_W(u) = \Phi_W(u+c/2,P) \qquad : \widehat{\mathcal{F}} \longrightarrow \widehat{\mathcal{F}}' \otimes W_z, \tag{6.1}$$

$$\widehat{\Psi}^*_W(u) = \Psi^*_W(u,P)e^{h^{(1)}Q/2} \qquad : W_z \otimes \widehat{\mathcal{F}} \longrightarrow \widehat{\mathcal{F}}'. \tag{6.2}$$

From the commutation relation of P and Q, the new operators $\widehat{\Phi}_W(u)$ and $\widehat{\Psi}^*_W(u)$ satisfy the following "intertwining relations".

$$\widehat{\Phi}^{(3)}_W(u_2)\widehat{L}^{+(1)}_V(u_1) = R^{+(1,3)}_{VW}(u_1-u_2,P+h/2)\widehat{L}^{+(1)}_V(u_1)\widehat{\Phi}^{(3)}_W(u_2), \tag{6.3}$$

$$\widehat{L}^{+(1)}_V(u_1)\widehat{\Psi}^{*(2)}_W(u_2) = \widehat{\Psi}^{*(2)}_W(u_2)\widehat{L}^{+(1)}_V(u_1)R^{+*(1,2)}_{VW}(u_1-u_2,P-(h^{(1)}+h^{(2)})/2). \tag{6.4}$$

Now let us restrict ourselves to the vector representation $W=V\cong \mathbb{C}v_+\oplus\mathbb{C}v_0\oplus\mathbb{C}v_-$. In this case, the R-matrix $R^+_{VV}(u,P)$ is given by $R^+(u,P)$ in (2.18), and the L-operator $\widehat{L}^+_V(u,P)$ by $\widehat{L}^+(u,P)$ in (5.6). Let us set the components of the vertex operators $\Phi_j(u), \Psi^*_j(u), (j=\pm,0)$ by

$$\widehat{\Phi}\left(u-\frac{1}{2}\right) = \sum_{j=\pm,0}\Phi_j(u)\otimes v_j, \tag{6.5}$$

$$\widehat{\Psi}^*\left(u-\frac{c+1}{2}\right)(v_j\otimes\cdot) = \Psi^*_j(u), \tag{6.6}$$

and the matrix elements of the L-operator $\widehat{L}^+(u)$ by

$$\widehat{L}^+(u)v_j = \sum_{m=0,\pm} v_m L^+_{m,j}(u). \tag{6.7}$$

Let us investigate the relations (6.3) and (6.4) in detail. From the components $[\ (-,-),(j)], j=\pm,0$ of (6.3), we have

$$\Phi_-\left(u_2+\frac{1}{2}\right)L^+_{-,j}(u_1) = \rho^+(u_1-u_2)L^+_{-,j}(u_1)\Phi_-\left(u_2+\frac{1}{2}\right). \tag{6.8}$$

Putting the definition $L^+_{-,j}(u)=K^+_-(u)E^+_{-,j}(u)$ into the above, we have

$$\Phi_-\left(u_2+\frac{1}{2}\right)K^+_-(u_1) = \rho^+(u_1-u_2)K^+_-(u_1)\Phi_-\left(u_2+\frac{1}{2}\right), \tag{6.9}$$

$$[\Phi_-(u_1),E^+_{-,0}(u_2)] = 0, \tag{6.10}$$

$$[\Phi_-(u_1),E^+_{-,+}(u_2)] = 0. \tag{6.11}$$

We have the sufficient condition of (6.10), (6.11).

$$\Phi_-(u_1)E(u_2) = E(u_2)\Phi_-(u_1), \quad [\Phi_-(z_1), P] = 0. \tag{6.12}$$

From the component $[(0,-),(-)]$ of (6.3), we have

$$\begin{aligned}\Phi_-\left(u_2+\frac{1}{2}\right)F^+_{0,-}(u_1)K^+_-(u_1) &= \rho^+(u)\bar{R}^{0-}_{0-}(u,P+h/2)F^+_{0,-}(u_1)K^+_-(u_1)\Phi_-\left(u_2+\frac{1}{2}\right)\\ &+ \rho^+(u)\bar{R}^{-0}_{0-}(u,P+h/2)K^+_-(u_1)\Phi_0\left(u_2+\frac{1}{2}\right).\end{aligned} \tag{6.13}$$

Let us assume the operator product $K^+_-(u_1)\Phi_-\left(u_2+\frac{1}{2}\right)$ has no pole at $u_1-u_2=-1-r$. Later we will check that, for $c=1$, this assumption is satisfied in a free field realization. Then from (6.9), we conclude the operator product $\Phi_-\left(u_2+\frac{1}{2}\right)K^+_-(u_1)$ has zero at $u_1-u_2=-1-r$. Therefore setting $u_1-u_2=-1-r$ in (6.13), we have

$$0 = \frac{[P+h/2+1/2]_+}{[P+h/2-1/2]_+}F^+_{0,-}(u_1)K^+_-(u_1)\Phi_-\left(u_2+\frac{1}{2}\right) + K^+_-(u_1)\Phi_0\left(u_2+\frac{1}{2}\right). \tag{6.14}$$

Then we have

$$\Phi_0(u) = F^+_{0,-}\left(u-r-\frac{1}{2}\right)\Phi_-(u). \tag{6.15}$$

Substituting (6.8) and (6.15) into (6.13), we get

$$\Phi_-(u_1)F(u_2) = -\frac{[u_1-u_2+1/2]}{[u_1-u_2-1/2]}F(u_2)\Phi_-(u_1). \tag{6.16}$$

Similarly, in order to investigate the structure of the component $\Phi_+(u)$, we have, from the components of $[(+,-),(-)]$ of (6.3),

$$\begin{aligned}&\Phi_-\left(u_2+\frac{1}{2}\right)F^+_{+,-}(u_1)K^+_-(u_1)\\ &= \rho^+(u)\bar{R}^{+-}_{+-}(u,P+h/2)F^+_{+,-}(u_1)K^+_-(u_1)\Phi_-\left(u_2+\frac{1}{2}\right)\\ &+\rho^+(u)\bar{R}^{00}_{+-}(u,P+h/2)F^+_{0,-}(u_1)K^+_-(u_1)\Phi_0\left(u_2+\frac{1}{2}\right)\\ &+\rho^+(u)\bar{R}^{-+}_{+-}(u,P+h/2)K^+_-(u_1)\Phi_+\left(u_2+\frac{1}{2}\right).\end{aligned} \tag{6.17}$$

On the other hands, from the component $[(+,-),(-,-)]$ of RLL relation (5.2), we have

$$\begin{aligned}K^+_-(u_2)F^+_{+,-}(u_1)K^+_-(u_2)^{-1} &= R^{+-}_{+-}(u)F^+_{+,-}(u_1) + R^{00}_{+-}(u)F^+_{0,-}(u_1)K^+_-(u_1)F^+_{0,-}(u_2)K^+_-(u_1)^{-1}\\ &+R^{-+}_{+-}(u)K^+_-(u_1)F^+_{+,-}(u_2)K^+_-(u_1)^{-1}.\end{aligned} \tag{6.18}$$

Putting the above into (6.17), we get

$$
\begin{aligned}
&\Phi_-\left(u_2+\frac{1}{2}\right)F^+_{+,-}(u_1)K^+_-(u_1)\\
&=\rho^+(u)\bar{R}^{-+}_{+-}(u|P+h/2)K_-(u_1)\Phi_+\left(u_2+\frac{1}{2}\right)\\
&\quad+\rho^+(u)K^+_-(u_2-r)F^+_{+,-}(u_1)K^+_-(u_2-r)^{-1}K^+_-(u_1)\Phi_-\left(u_2+\frac{1}{2}\right)\\
&\quad+\rho^+(u)\bar{R}^{-+}_{+-}(u|P+h/2)K^+_-(u_1)F^+_{+,-}(u_2-r)\Phi_-\left(u_2+\frac{1}{2}\right).
\end{aligned}
\tag{6.19}
$$

Note that at the point $u_1-u_2=-1-r$, $\rho^+(u)$ has a zero, but $\rho^+(u)\bar{R}^{-+}_{+-}(u|P+h/2)$ have no zeros. In addition, under the same assumption given just below (6.13), the product $\Phi_-\left(u_2+\frac{1}{2}\right)K^+_-(u_1)$ vanishes at $u_1-u_2=-1-r$. Setting $u_1-u_2=-1-r$ in (6.19), we thus have the following formula for $\Phi_+(z)$.

$$
\Phi_+(u) \quad = \quad -F^+_{+,-}\left(u-r-\frac{1}{2}\right)\Phi_-(u). \tag{6.20}
$$

In the next section, we construct a free field realization of the type-I vertex operators using the relations (6.9), (6.12), (6.15), (6.16) and (6.20) for $c=1$. We can check that the resultant vertex operators satisfy the intertwining relation (6.3).

Similarly, the sufficient conditions for the type-II vertex operators are extracted from (6.4) as follows.

$$
\Psi^*_-\left(u_2+\frac{1+c}{2}\right)K_-(u_1)\rho^{+*}(u)=K^+_-(u_1)\Psi^*_-\left(u_2+\frac{1+c}{2}\right), \tag{6.21}
$$

$$
\Psi^*_-(u_1)F(u_2)=F(u_2)\Psi^*_-(u_1),\quad [\Psi^*_-(u),P+h/2]=0, \tag{6.22}
$$

$$
\Psi^*_-(u_1)E(u_2)=-\frac{[u_1-u_2-\frac{1}{2}]^*}{[u_1-u_2+\frac{1}{2}]^*}E(u_2)\Psi^*_-(u_1), \tag{6.23}
$$

$$
\Psi^*_+(u)=-\Psi^*_-(u)E_{-,+}\left(u-\frac{1+c}{2}-r^*\right), \tag{6.24}
$$

$$
\Psi^*_0(u)=\Psi^*_-(u)E_{-,0}\left(u-\frac{1+c}{2}-r^*\right). \tag{6.25}
$$

6.2 Free Field Realizations

Now we construct a free field realization of the vertex operators fixing the representation level $c=1$. For this purpose, we introduce the simple root operator α, defined by

$$
[h,\alpha]=2,\ [a_m,\alpha]=0,\ [P,\alpha]=0,\ [Q,\alpha]=0,\ [\alpha,\bar{\alpha}]=0. \tag{6.26}
$$

If we introduce $\hat{\alpha}$ by

$$\hat{\alpha} = \alpha + \bar{\alpha}, \tag{6.27}$$

we have

$$[h, \hat{\alpha}] = 2, \ [a_m, \hat{\alpha}] = 0, \ [P, \hat{\alpha}] = 0, \ [Q, \hat{\alpha}] = \pi i. \tag{6.28}$$

Then the following statement holds.

Proposition 6.1 *For $c = 1$, we have the free field realization of the currents $E(z)$ and $F(z)$.*

$$\begin{aligned} E(z) &= \epsilon(q) : \exp\left(-\sum_{m\neq 0} \frac{1}{[m]_q} \alpha_m z^{-m}\right) : e^{\hat{\alpha}} z^{h/2+1/2} e^{-Q} z^{-P/r^*}, && (6.29) \\ F(z) &= \epsilon(q) : \exp\left(\sum_{m\neq 0} \frac{1}{[m]_q} \beta_m z^{-m}\right) : e^{-\hat{\alpha}} z^{-h/2+1/2} z^{P/r+h/2r}. && (6.30) \end{aligned}$$

Here we have set

$$\epsilon(q) = (q^{1/2} + q^{-1/2})^{-1/2}. \tag{6.31}$$

Together with free field realizations of $K(z)$ (3.55), we get a free field realization of the level one elliptic algebra $U_{q,p}(A_2^{(2)})$.

Now substituting the free field realization of $E(z),\ F(z),\ K(z)$ into (3.53)- (3.55), we obtain a realization of the half currents and the L-operator $\widehat{L}^+(u)$ satisfying the RLL-relation (5.2) for $c = 1$. Using this L-operator in the "intertwining relations", (6.9), (6.12), (6.15), (6.16), (6.20), for type I and (6.21)-(6.25) for the type II, one can solve them for the vertex operators. The results are stated as follows.

Theorem 6.2 *The highest components of the type-I and type-II vertex operators $\Phi_-(u)$ and $\Psi^*_-(u)$ are realized in terms of the free field by*

$$\begin{aligned} \Phi_-(z) &= : \exp\left(-\sum_{m\neq 0} \frac{1}{[2m]_q - [m]_q} \beta_m z^{-m}\right) : e^{\hat{\alpha}} z^{h/2+1/2} z^{-P/r-h/2r-1/r}, && (6.32) \\ \Psi^*_-(z) &= : \exp\left(\sum_{m\neq 0} \frac{[rm]_q}{[2m]_q - [m]_q} \alpha_m z^{-m}\right) : e^{-\hat{\alpha}} z^{-h/2+1/2} e^{Q} z^{P/r^*+1/r^*}. && (6.33) \end{aligned}$$

For the other components of type-I vertex operator $\Phi_j(u)(j=\pm,0)$, we get the following, by using (6.15) and (6.20).

$$\begin{aligned}\Phi_0(u) &= a_{0,-}\oint_{C_0}\frac{dz'}{2\pi i z'}\Phi_-(u)F(z')\frac{[u-u'+P+\frac{h}{2}]_+}{[u-u'+\frac{1}{2}][P+\frac{h}{2}+\frac{1}{2}]_+}\\ &= -a_{0,-}\oint_{C_0}\frac{dz'}{2\pi i z'}F(z')\Phi_-(u)\frac{[u-u'+P+\frac{h}{2}]_+}{[u-u'-\frac{1}{2}][P+\frac{h}{2}+\frac{1}{2}]_+}.\end{aligned}\tag{6.34}$$

Here the contour C_0 is specified by the condition.

$$C_0: \quad |q^{-1}z|<|z'|<|p^{-1}q^{-1}z|.$$

The component $\Phi_+(u)$ is given by

$$\begin{aligned}\Phi_+(u) &= -a_{+,-}\oint\oint_{C_+}\frac{dz'}{2\pi i z'}\frac{dz''}{2\pi i z''}\Phi_-(u)F(z')F(z'')\frac{[P+\frac{h}{2}]_+}{[P+\frac{h}{2}-\frac{1}{2}]_+[P+\frac{h}{2}-1]_+[2P+h]}\\ &\times \frac{[u-u'+2P+h-\frac{3}{2}][u'-u''+P+\frac{h}{2}]_+}{[u-u'+\frac{1}{2}][u'-u''+\frac{1}{2}]}\\ &= -a_{+,-}\oint\oint_{C^+}\frac{dz'}{2\pi i z'}\frac{dz''}{2\pi i z''}F(z')F(z'')\Phi_-(u)\frac{[P+\frac{h}{2}]_+}{[P+\frac{h}{2}-\frac{1}{2}]_+[P+\frac{h}{2}-1]_+[2P+h]}\\ &\times \frac{[u-u'+2P+h-\frac{3}{2}][u'-u''+P+\frac{h}{2}]_+[u-u''+\frac{1}{2}]}{[u-u'-\frac{1}{2}][u-u''-\frac{1}{2}][u'-u''+\frac{1}{2}]}.\end{aligned}\tag{6.35}$$

The contour C_+ is specified by

$$C_+: \quad |q^{-1}z|<|z'|<|p^{-1}q^{-1}z|, \quad |q^{-1}z|<|z''|<|p^{-1}q^{-1}z|, \quad |pqz'|<|z''|<|qz'|.$$

Similarly, for Type-II vertex operators, the component $\Psi_0^*(u)$ is given by

$$\begin{aligned}\Psi_0^*(u) &= a_{-,0}^*\oint_{C_0^*}\frac{dz'}{2\pi i z'}\Psi_-^*(u)E(z')\frac{[u-u'-P]_+^*}{[u-u'-\frac{1}{2}]^*[P-\frac{1}{2}]_+^*}\\ &= -a_{-,0}^*\oint_{C_0^*}\frac{dz'}{2\pi i z'}E(z')\Psi_-^*(u)\frac{[u-u'-P]_+^*}{[u-u'+\frac{1}{2}]^*[P-\frac{1}{2}]_+^*}.\end{aligned}\tag{6.36}$$

The contour C_0^* satisfies

$$C_0^*: \quad |q^{-1}z|<|z'|<|qz|.$$

The component $\Psi_0^*(u)$ is given by

$$\Psi_+^*(u) = -a_{-,+}^*\oint\oint_{C_+^*}\frac{dz'}{2\pi i z'}\frac{dz''}{2\pi i z''}\Psi_-^*(u)E(z')E(z'')\frac{1}{[P-\frac{1}{2}]_+^*[2P-2]}$$

$$\times \quad \frac{[u-u'-2P+\frac{3}{2}]^*[u'-u''-P]^*_+}{[u-u'-\frac{1}{2}]^*[u'-u''-\frac{1}{2}]^*}$$
$$= \quad -a^*_{-,+}\oint\oint_{C^*_+}\frac{dz'}{2\pi i z'}\frac{dz''}{2\pi i z''}E(z')E(z'')\Psi^*_-(u)\frac{1}{[P-\frac{1}{2}]^*_+[2P-2]}$$
$$\times \quad \frac{[u-u'-2P+\frac{3}{2}]^*[u'-u''-P]^*_+[u-u''-\frac{1}{2}]^*}{[u-u'+\frac{1}{2}]^*[u-u''+\frac{1}{2}]^*[u'-u''-\frac{1}{2}]^*}. \tag{6.37}$$

Here the contour C^*_+ is specified by the condition

$$C^*_+ : \quad |q^{-1}z|<|z'|<|qz|, \quad |q^{-1}z|<|z''|<|qz|, \quad |q^{-1}z'|<|z''|<|qz'|.$$

Remark The free field realizations of the vertex operators (6.32) -(6.37) are the same as those of the dilute A_L model obtained in [15], up to a gauge transformation.

In addition we can verify the following commutation relation.

Proposition 6.3 *The highest components $\Phi_-(u)$ and $\Psi^*_-(u)$ satisfy*

$$\Phi_-(u_1)\Psi^*_-(u_2)=\chi(u_1-u_2)\Psi^*_-(u_2)\Phi_-(u_1). \tag{6.38}$$

Here we have set

$$\chi(u)=-z^{-1}\frac{\Theta_{q^6}(qz)\Theta_{q^6}(q^2z)}{\Theta_{q^6}(q/z)\Theta_{q^6}(q^2/z)}. \tag{6.39}$$

6.3 Commutation Relations of the Vertex Operators

We next study the commutation relations of the vertex operators and show that our realization satisfies the full intertwining relations for $c=1$.

Theorem 6.4 *The free field realizations of the type-I vertex operator $\Phi_\mu(u)$ (6.32), (6.34), (6.35), and the type-II vertex operator $\Psi^*_\mu(u)$ (6.33), (6.36), (6.37), satisfy the following commutation relations.*

$$\Phi_{j_2}(u_2)\Phi_{j_1}(u_1)=\sum_{j_1',j_2'=\pm,0}R^{j_1'j_2'}_{j_1j_2}(u_1-u_2,P+h)\ \Phi_{j_1'}(u_1)\Phi_{j_2'}(u_2), \tag{6.40}$$

$$\Psi^*_{j_1}(u_1)\Psi^*_{j_2}(u_2)=\sum_{j_1',j_2'=\pm,0}\Psi^*_{j_2'}(u_2)\Psi^*_{j_1'}(u_1)\ R^{*j_1j_2}_{j_1'j_2'}(u_1-u_2,P), \tag{6.41}$$

$$\Phi_j(u_1)\Psi^*_k(u_2)=\chi(u_1-u_2)\ \Psi^*_k(u_2)\Phi_j(u_1). \tag{6.42}$$

Here we set

$$R(u, P+h) = \mu(u)\bar{R}(v, P+h), \qquad R^*(u, P) = \mu^*(u)\bar{R}^*(u, P), \tag{6.43}$$

with

$$\mu(u) = z^{\frac{1}{r}-1} \frac{\{pq^4z\}\{pq^3z\}\{q^3z\}\{q^2z\}\{pq/z\}\{p/z\}\{q^6/z\}\{q^5/z\}}{\{pq^4/z\}\{pq^3/z\}\{q^3/z\}\{q^2/z\}\{pqz\}\{pz\}\{q^6z\}\{q^5z\}}. \tag{6.44}$$

and $\mu^*(u) = \mu(u)|_{r\to r^*}$. *Here* $\chi(u)$ *is given by (6.39).*

The proof is the similar as those of Theorem 4.2.

Now let us investigate the intertwining relation for level $c = 1$. For this purpose, we construct a L-operator as a composition of type I and II vertex operators [17].

Theorem 6.5 *For* $c = 1$, *the components of the L-operator* $\widehat{L}^+(u)$ *(5.1) is given by the following product of the type-I and type-II vertex operators.*

$$L^+_{j,k}(u) = g^{-1}\Psi^*_k(u+r)\Phi_j(u+r+1/2), \quad (j, k = \pm, 0). \tag{6.45}$$

Here we set

$$g = -\frac{(pq^6;q^6)_\infty(pq^5;q^6)_\infty}{(pq^3;q^6)_\infty(pq^2;q^6)_\infty}\left(\frac{\{q^2p\}\{q^3p\}\{q^3p\}\{q^4p\}}{\{p\}\{qp\}\{q^5p\}\{q^6p\}} \times (p \leftrightarrow p^*)^{-1}\right). \tag{6.46}$$

The proof is similar to the one of Theorem 6.5 in [8].

Remark By using the commutation relations of the vertex operators (6.40)-(6.42) and the formula

$$\frac{\rho^+(u)}{\rho^{+*}(u)} = \frac{\mu(u)\chi(\frac{1}{2}-u)}{\mu^*(u)\chi(\frac{1}{2}+u)}, \tag{6.47}$$

one can prove the $RLL = LLR^*$ relation (5.2) for $c = 1$ directly.

In the same way, one can verify the "intertwining relations" (6.3) and (6.4) of vertex operators.

Corollary 6.6 *For* $c = 1$, *the type-I and the type II vertex operators* $\widehat{\Phi}_V(u)$, $\widehat{\Psi}^*_V(u)$ *satisfy the full intertwining relations (6.3) and (6.4) with* $V = W \cong \mathbb{C}^3$.

7 Discussion

Extending the construction of the elliptic algebra to the twisted affine Lie algebra case, we have derived the elliptic algebra $U_{q,p}(A_2^{(2)}),\ p = q^{2r}$ and shown that it provides the Drinfeld realization of the elliptic quantum group $\mathcal{B}_{q,\lambda}(A_2^{(2)})$. Based on this, we have derived the type I and II vertex operators of $U_{q,p}(A_2^{(2)})$ and identified them with the vertex operators in the dilute A_L model with $r = 2\frac{L+1}{L+2}$. Our result thus gives a representation theoretical foundation to the work [15].

There are some open problems.

(i) We here studied the simplest twisted elliptic quantum group $\mathcal{B}_{q,\lambda}(A_2^{(2)})$ and associated elliptic algebra $U_{q,p}(A_2^{(2)})$. To generalize our consideration to the higher rank cases associated with $A_{2n}^{(2)}$ and $A_{2n+1}^{(2)}$, or moreover to other types of affine Lie algebras, is an interesting problem.

(ii) Our realization of the elliptic algebra $U_{q,p}(A_2^{(2)})$ based on the Drinfeld currents of $U_q(A_2^{(2)})$ and the Heisenberg algebra $\mathbb{C}\{\mathcal{H}\}$ is valid for a generic level c. In order to perform an algebraic analysis of the solvable lattice models, a free field realization is useful. For example, to consider a fusion of the dilute A_L model, i.e. a higher spin extension, we need a free field realization (Wakimoto construction) of the elliptic algebra $U_{q,p}(A_2^{(2)})$ in higher level.

(iii) The Wakimoto realization of the affine quantum group $U_q(A_2^{(2)})$ itself is interesting. It should be used to solve the q-KZ equation as well as the q-difference equation for the twistor $F(r,s)$, which we have solved partly (see Appendix B). The same thing is true for the other types of affine Lie algebra and should lead us to a proof of the conjecture on the connection matrix of the q-KZ equation given by Frenkel and Reshetikhin[18].

(iv) It is known in some cases that the generating functions of the q-deformed Virasoro or W- algebras can be obtained from a fusion of the vertex operators of corresponding elliptic algebra $U_{q,p}(\mathfrak{g})$ [19, 20, 15, 21]. It is interesting to examine the same procedure in various $U_{q,p}(\mathfrak{g})$ and derive corresponding q-W algebras. The results should be compared with those in [22].

(v) It is also an interesting problem to investigate the scaling limit of the half currents and the L-operators of $U_{q,p}(A_2^{(2)})$ and derive the vertex operators [19, 23]. The result should be applied to the Izergin-Korepin model[24] in the massless regime where a generic form of the correlation functions was studied in [25]. The type-II vertex operators should provide the Zamolodchikov-Faddeev algebra for the $A_2^{(2)}$ Toda field theory with imaginary coupling constant, and enable us to derive the soliton S-matrix.

We hope to report on some of the issues listed here in the near future.

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A Finite Dimensional Representation

The evaluation module (π_w, V_w) in terms of the Drinfeld generators, is defined by the following formulae.

$$\pi_w(h) = 2(E_{++} - E_{--}), \quad \pi_w(c) = 0, \tag{A.1}$$

$$\pi_w(a_m) = \frac{[m]_q}{m}(w/q)^m \left(q^{-m}E_{++} + (1-q^m)E_{00} - q^{2m}E_{--}\right), \tag{A.2}$$

$$\pi_w(x_k^+) = (w/q)^k(aE_{+0} + q^k bE_{0-}), \tag{A.3}$$

$$\pi_w(x_k^-) = (w/q)^k(q^k b^{-1}E_{-0} + a^{-1}E_{0+}). \tag{A.4}$$

Here we have used $E_{++} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $E_{00} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $E_{--} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

In what follows we set $a = b = 1$. We have

$$\pi_w(x^+(z)) = E_{+0}\delta(w/qz) + E_{0-}\delta(w/z), \tag{A.5}$$

$$\pi_w(x^-(z)) = E_{-0}\delta(w/z) + E_{0+}\delta(w/qz). \tag{A.6}$$

$$\begin{aligned}
\pi_w(u^+(z,p)) &= \frac{(pq^3z/w;p)_\infty}{(pqz/w;p)_\infty}E_{++} \\
&+ \frac{(pq^2z/w;p)_\infty(pq^{-1}z/w;p)_\infty}{(pqz/w;p)_\infty(pz/w;p)_\infty}E_{00} + \frac{(pq^{-2}z/w;p)_\infty}{(pz/w;p)_\infty}E_{--}, &(A.7)\\
\pi_w(u^-(z,p)) &= \frac{(pq^{-3}w/z;p)_\infty}{(pq^{-1}w/z;p)_\infty}E_{++} \\
&+ \frac{(pqw/z;p)_\infty(pq^{-2}w/z;p)_\infty}{(pw/z;p)_\infty(pq^{-1}w/z;p)_\infty}E_{00} + \frac{(pq^2w/z;p)_\infty}{(pw/z;p)_\infty}E_{--}. &(A.8)
\end{aligned}$$

Let us calculate finite dimensional representation of the elliptic current.

$$\pi_w(e(z,p)) = \frac{(pq^3z/w;p)_\infty}{(pqz/w;p)_\infty}E_{+0}\delta(w/qz) + \frac{(pq^2z/w;p)_\infty(pq^{-1}z/w;p)_\infty}{(pqz/w;p)_\infty(pz/w;p)_\infty}E_{0-}\delta(w/z), \tag{A.9}$$

$$\pi_w(f(z,p)) = \frac{(pqw/z;p)_\infty(pq^{-2}w/z;p)_\infty}{(pw/z;p)_\infty(pq^{-1}w/z;p)_\infty}E_{-0}\delta(w/z) + \frac{(pq^{-3}w/z;p)_\infty}{(pq^{-1}w/z;p)_\infty}E_{0+}\delta(w/qz), \tag{A.10}$$

$$\begin{aligned}
\pi_w(k(z,p)) &= \rho^+(q^{-r+2}z/w) \\
&\times \left(E_{++} + \frac{\Theta_p(q^rz/w)}{\Theta_p(q^{r+2}z/w)}E_{00} + \frac{\Theta_p(q^rz/w)\Theta_p(q^{r-1}z/w)}{\Theta_p(q^{r+2}z/w)\Theta_p(q^{r+1}z/w)}E_{--}\right), &(A.11)\\
\pi_w(\psi(z,p)) &= \frac{\Theta_p(q^{r+3}z/w)}{\Theta_p(q^{r+1}z/w)}E_{++} \\
&+ \frac{\Theta_p(q^{r+2}z/w)\Theta_p(q^{r-1}z/w)}{\Theta_p(q^{r+1}z/w)\Theta_p(q^rz/w)}E_{00} + \frac{\Theta_p(q^{r-2}z/w)}{\Theta_p(q^rz/w)}E_{--}. &(A.12)
\end{aligned}$$

B Twistor

We here consider the difference equations of the twistor $F(\lambda)$ for $\mathcal{B}_{q,\lambda}(A_2^{(2)})$. The general framework was given in [3]. Let us consider the case of the affine algebra $A_2^{(2)}$. Taking a basis $\{c, d, \alpha_1^\vee\}$ of the Cartan subalgebra $\mathfrak{h}$ of $A_2^{(2)}$. We parametrize the dynamical variable λ as

$$\lambda - \rho = r^*d + s'c + \frac{1}{2}\left(s + \frac{r\tau}{2}\right)\alpha_1^\vee, \qquad \left(r^* = r - c,\ \tau = -\frac{2\pi i}{\log q^{2r}}\right), \tag{B.1}$$

where $\rho = 3d + \frac{1}{4}\alpha_1^\vee$ is the Weyl vector. Let us set

$$\mathcal{R}(z) = \mathrm{Ad}(z^d \otimes 1)(\mathcal{R}), \tag{B.2}$$

$$F(z,p,w) = \mathrm{Ad}(z^d \otimes 1)(F(\lambda)), \tag{B.3}$$

$$\mathcal{R}(z;p,w) = \mathrm{Ad}(z^d \otimes 1)(\mathcal{R}(\lambda)) = \sigma(F(z^{-1};p,w))\mathcal{R}(z)F(z;p,w)^{-1}, \tag{B.4}$$

where $w = q^{2(s+\frac{r\tau}{2})}$. In particular, for $z=0$, $q^{c\otimes d+d\times c}\mathcal{R}(0)$ reduces to the universal R matrix of $U_q(A_1)$. From [3], we have the difference equation for the twistor.

$$F(pq^{2c^{(1)}}z;p,w) = (\bar{\varphi}_w^{-1}\otimes id)(F(z;p,w))q^T\mathcal{R}(pq^{2c^{(1)}}z), \tag{B.5}$$

$$F(0;p,w) = F_{A_1}(w), \tag{B.6}$$

where $\bar{\varphi}_w = \mathrm{Ad}(q^{\alpha_1^{\vee 2}/4}w^{\alpha_1^\vee/2})$ and $T = \frac{1}{2}c\otimes d + \frac{1}{2}d\otimes c + \frac{1}{4}\alpha_1^\vee\otimes\alpha_1^\vee$.

We are interested in the vector representation $(\pi_z, V), V=\mathbb{C}^3$ given in Appendix A. We set

$$F_{VV}(z;p,w) = (\pi_1\otimes\pi_1)F(z;p,w) = (\pi_{z_1}\otimes\pi_{z_2})(F(\lambda)), \tag{B.7}$$

$$R_{VV}(z;p,w) = (\pi_1\otimes\pi_1)\mathcal{R}(z;p,w) = (\pi_{z_1}\otimes\pi_{z_2})(\mathcal{R}(\lambda)), \tag{B.8}$$

$$R_{VV}(z) = (\pi_1\otimes\pi_1)\mathcal{R}(z) = (\pi_{z_1}\otimes\pi_{z_2})\mathcal{R} \tag{B.9}$$

where $z = z_1/z_2$. The trigonometric R-matrix $R_{VV}(z)$ is given as follows.

$$R_{VV}(z) = \rho_{VV}(z)\bar{R}_{VV}(z), \tag{B.10}$$

$$\bar{R}_{VV}(z) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & b(z) & 0 & c(z) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & d(z) & 0 & e(z) & 0 & f(z) & 0 & 0 \\ 0 & z\,c(z) & 0 & b(z) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -q^2z\,e(z) & 0 & j(z) & 0 & e(z) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & b(z) & 0 & c(z) & 0 \\ 0 & 0 & z\,n(z) & 0 & -q^2z\,e(z) & 0 & d(z) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & z\,c(z) & 0 & b(z) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$b(z) = -\frac{q(1-z)}{1-q^2z}, \qquad c(z) = \frac{1-q^2}{1-q^2z}, \qquad d(z) = \frac{(1-z)q^2(1-qz)}{(1-q^2z)(1-q^3z)},$$

$$e(z) = \frac{i(1-q^2)q^{\frac{1}{2}}(1-z)}{(1-q^2z)(1-q^3z)}, \qquad f(z) = \frac{(1-q^2)(1+q-q^3z-qz)}{(1-q^2z)(1-q^3z)},$$

$$j(z) = -\frac{q(1-z)}{1-q^2z} + \frac{(1-q^2)(1-q^3)z}{(1-q^2z)(1-q^3z)}, \qquad n(z) = \frac{(1-q^2)(1+q^2-q^3z-q^2z)}{(1-q^2z)(1-q^3z)}.$$

The function $\rho_{VV}(z)$ is given by

$$\rho_{VV}(z) \;=\; q^{-1}\frac{(1/z;q^6)_\infty(q/z;q^6)_\infty(q^5/z;q^6)_\infty(q^6/z;q^6)_\infty}{(q^2/z;q^6)_\infty(q^3/z;q^6)_\infty^2(q^4/z;q^6)_\infty}. \tag{B.11}$$

Noting $\pi_1(c)=0$, we have the difference equation

$$F_{VV}(pz;p,w)^t = R_{VV}(pz)^t K (D_w \otimes 1)^{-1} F_{VV}(z;p,w)^t (D_w \otimes 1), \tag{B.12}$$

where X^t means the transpose of X, and we have set

$$\begin{aligned} K &= \mathrm{Diag}(q,1,q^{-1},1,1,1,q^{-1},1,q), && \text{(B.13)} \\ D_w &= \mathrm{Diag}(q^{-1}w^{-1},1,q^{-1}w). && \text{(B.14)} \end{aligned}$$

From the from of $R_{VV}(z)$, one can set

$$F(z;p,w) = f(z)\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & X_{11}^{(+)}(z) & 0 & X_{12}^{(+)}(z) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & Y_{11}(z) & 0 & Y_{12}(z) & 0 & Y_{13}(z) & 0 & 0 \\ 0 & X_{21}^{(+)}(z) & 0 & X_{22}^{(+)}(z) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & Y_{21}(z) & 0 & Y_{22}(z) & 0 & Y_{23}(z) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & X_{11}^{(-)}(z) & 0 & X_{12}^{(-)}(z) & 0 \\ 0 & 0 & Y_{31}(z) & 0 & Y_{32}(z) & 0 & Y_{33}(z) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & X_{21}^{(-)}(z) & 0 & X_{22}^{(-)}(z) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then the q-difference equation (B.12) is equivalent to the following equations.

$$f(pz) = q\rho_{VV}(pz) f(z), \tag{B.15}$$

$$\begin{pmatrix} X_{11}^{(\pm)}(pz) & X_{12}^{(\pm)}(pz) \\ X_{21}^{(\pm)}(pz) & X_{22}^{(\pm)}(pz) \end{pmatrix} = q^{-1} \begin{pmatrix} X_{11}^{(\pm)}(z) & q^{\pm 1} w X_{12}^{(\pm)}(z) \\ q^{\mp 1} w^{-1} X_{21}^{(\pm)}(z) & X_{22}^{(\pm)}(pz) \end{pmatrix} \begin{pmatrix} b(pz) & c(pz) \\ pz\, c(pz) & b(pz) \end{pmatrix}, \tag{B.16}$$

$$\begin{aligned} &\begin{pmatrix} Y_{11}(pz) & Y_{12}(pz) & Y_{13}(pz) \\ Y_{21}(pz) & Y_{22}(pz) & Y_{23}(pz) \\ Y_{31}(pz) & Y_{32}(pz) & Y_{33}(pz) \end{pmatrix} \\ &= q^{-2} \begin{pmatrix} Y_{11}(z) & wY_{12}(z) & w^2 Y_{13}(pz) \\ qw^{-1}Y_{21}(z) & qY_{22}(z) & qwY_{23}(z) \\ w^{-2}Y_{31}(z) & w^{-1}Y_{32}(z) & Y_{33}(z) \end{pmatrix} \begin{pmatrix} d(pz) & e(pz) & f(pz) \\ -q^2 pz\, e(pz) & j(pz) & e(pz) \\ pz\, n(pz) & -q^2 pz\, e(pz) & d(pz) \end{pmatrix}. \end{aligned} \tag{B.17}$$

The two 2×2 matrix equations for $X_{ij}^{(\pm)}(z)$ are the same as the one appeared in the $\widehat{\mathfrak{sl}}_2$ case [3], if we change $b(z)$ to $-b(z)$ and make the following identification.

$$q^{\pm 1} w = w_{\mathfrak{sl}_2}^{-1} \quad \text{i.e.} \quad -s_{\mathfrak{sl}_2} = s + \frac{r\tau}{2} \pm \frac{1}{2},$$

where $w_{\mathfrak{sl}_2}$ and $s_{\mathfrak{sl}_2}$ denote w and s in [3], respectively. Hence from the elliptic R matrix for $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_2)$ ((4.18) in [3]), we determine the following parts of our elliptic R-matrix

$$\begin{pmatrix} R_{+0}^{+0} & R_{+0}^{0+} \\ R_{0+}^{+0} & R_{0+}^{0+} \end{pmatrix} = \begin{pmatrix} -\frac{[s+3/2]_+[s-1/2]_+}{[s+1/2]_+^2}\frac{[u]}{[u+1]} & e^{\frac{\pi i u}{r}}\frac{[1][s+1/2-u]_+}{[s+1/2]_+[u+1]} \\ e^{-\frac{\pi i u}{r}}\frac{[1][s+1/2+u]_+}{[s+1/2]_+[u+1]} & -\frac{[u]}{[u+1]} \end{pmatrix}, \tag{B.18}$$

$$\begin{pmatrix} R_{0-}^{0-} & R_{0-}^{-0} \\ R_{-0}^{0-} & R_{-0}^{-0} \end{pmatrix} = \begin{pmatrix} -\frac{[s-3/2]_+[s+1/2]_+}{[s-1/2]_+^2}\frac{[u]}{[u+1]} & e^{\frac{\pi i u}{r}}\frac{[1][s-1/2-u]_+}{[s-1/2]_+[u+1]} \\ e^{-\frac{\pi i u}{r}}\frac{[1][s-1/2+u]_+}{[s-1/2]_+[u+1]} & -\frac{[u]}{[u+1]} \end{pmatrix}. \tag{B.19}$$

By a gauge transformation, these yields the corresponding matrix elements in (2.19).

As for the 3×3 part, we have no known solutions. The Wakimoto realization of $U_q(A_2^{(2)})$ should be useful to solve the q-KZ equation for the intertwining operators (vertex operators) of $U_q(A_2^{(2)})$ as well as (B.17).

C Proof of the Relation (4.27)

Let us set

$$h(v) = -\frac{[v+1]^*[v-1/2]^*}{[v-1]^*[v+1/2]^*}. \tag{C.1}$$

In the integrand of the half current $E^+_{-,+}(u)$ (4.9), we call $E(z')E(z'')$ the operator part, and the ratio of the product of the theta functions the coefficient part. We keep coefficient parts in the right of operator parts. According to the relation (3.62), we have the equality

$$\oint \frac{dz'}{2\pi i z'}\frac{dz''}{2\pi i z''}E(z')E(z'')A(u,u'') = \oint \frac{dz'}{2\pi i z'}\frac{dz''}{2\pi i z''}E(z')E(z'')h(u''-u')A(u'',u'),$$

when the integration contours for z' and z'' are the same. Here we set $z'=q^{2u'},\ z''=q^{2u''}$. We define 'weak equality' in the following sense[**?**]. The two coefficient functions $A(u',u'')$ and $B(u',u'')$ coupled to $E(z')E(z'')$ in integrals are equal in weak sense if

$$A(u',u'')+h(u''-u')A(u'',u') = B(u',u'')+h(u''-u')B(u'',u').$$

We write the weak equality as

$$A(u',u'') \sim B(u',u'').$$

To prove the equality (4.27), it is enough to show the equalities of coefficient parts in weak sense.

Setting $z_i = q^{2u_i}$ $(i = 1, 2)$ and $u = u_1 - u_2$, let us consider RHS-LHS of (4.27) given as follows.

$$\oint \frac{dz'}{2\pi i z'} \oint \frac{dz''}{2\pi i z''} E(z')E(z'')F(u_1, u_2, u', u'', L),$$

where

$$\begin{aligned}
&F(u_1, u_2, u', u'', L) \\
&= \frac{[u_2 - u' - 2P + 2 + c/2]^*[u' - u'' - P]^*_+[1 + u]^*[u + 3/2]^*[1]^{*2}}{[u_2 - u' + c/2]^*[u' - u'' - 1/2]^*[u]^*[2P - 2]^*[P - 1/2]^*_+[u + 1/2]^*} \\
&+ \frac{[u_2 - u' - P + (c+1)/2]^*_+[u_1 - u' + 1 + c/2]^*[u_1 - u" - P + (c+1)/2]^*_+[u + P + 1]^*_+[1]^*}{[u_2 - u' + c/2]^*[u_1 - u' + c/2]^*[u_1 - u'' + c/2]^*[P - 1/2]^{*2}_+[P + 1/2]^*_+[u + 1/2]^*} \\
&- \frac{[u_1 - u' - 2P + 2 + c/2]^*[u' - u'' - P]^*_+[1]^{*2}}{[u_1 - u' + c/2]^*[u' - u'' - 1/2]^*[P - 1/2]^*_+[u + 1/2]^*} \\
&\quad\times \left(\frac{[u + 2P - 1]^*[1]^*[u + 3/2]^*}{[u]^*[2P - 1]^*[2P - 2]^*} + \frac{[P]^*_+[u + 2P + 1/2]^*[1]^*}{[2P]^*[2P - 1]^*[P - 1]^*_+} \right) \\
&- \frac{[u_2 - u' - 2P + c/2]^*[u' - u'' - P - 1]^*_+[u_1 - u' + 1 + c/2]^*[u_1 - u'' + 1 + c/2]^*[1]^{*2}}{[u_2 - u' + c/2]^*[u' - u'' - 1/2]^*[u_1 - u' + c/2]^*[u_1 - u'' + c/2]^*[P + 1/2]^*_+[2P]^*}.
\end{aligned}$$

We will show that $F(u_1, u_2, u', u'', L) \sim 0$. For this purpose, we consider the function of u' defined by

$$F(u') = F(u_1, u_2, u', u'', L) + h(u'' - u')F(u_1, u_2, u'', u', L).$$

Then it is not so hard to see that $F(u')$ is a quasi-periodic function having zeros at least at $u' = u''$ and $u' = u'' + 1$. The quasi-periodicity is given by

$$\begin{aligned}
F(u' + \tau^* r^*) &= -e^{-\frac{2\pi i}{r}(P-3/2)}F(u'), \\
F(u' + r^*) &= F(u').
\end{aligned}$$

Therefore if we set

$$G(u') = F(u')\frac{[u' - u'' - P + 3/2]^*}{[u' - u'']^*},$$

$G(u')$ is a doubly periodic function of u' and $G(u'' + 1) = 0$. It is then enough to show that $G(u')$ is an entire function.

In $G(u')$, some of terms have the first order poles at $u' = u_1 + c/2,\ u_2 + c/2, u'' + 1/2, u'' - 1$. We checked that all the residues of the function $G(u')$ at these poles vanish.

For example, at $u' = u_2 + c/2$ the residue is given by

$$\begin{aligned}\mathrm{Res}_{u'=u_2+c/2} G(u') \frac{dz'}{2\pi i z'} \quad = \quad & -\frac{[u+1]^*[u_2-u''-P+c/2]^*_+[u+3/2]^*[1]^{*2}}{[u]^*[u_2-u''-1/2+c/2]^*[P-1/2]^*_+[u+1/2]^*} \\ & + \frac{[u+1]^*[u_1-u''-P+(c+1)/2]^*_+[u+P+1]^*_+[1]^{*3}}{[u]^*[u_1-u''+c/2]^*[P+1/2]^*_+[P-1/2]^*_+[u+1/2]^*} \\ & + \frac{[u+1]^*[u_1-u''+1+c/2]^*[u_2-u''-P-1+c/2]^*_+[1]^{*2}}{[u]^*[u_1-u''+c/2]^*[u_2-u''-1/2+c/2]^*[P+1/2]^*_+}.\end{aligned}$$

One can apply the following theta function identity to combine the 1st and the 3rd terms.

$$\begin{aligned}&[u+x]^*[u-x]^*[v+y]^*_+[v-y]^*_+ - [u+y]^*[u-y]^*[v+x]^*_+[v-x]^*_+ \\ &\quad = -[x-y]^*[x+y]^*[u+v]^*_+[u-v]^*_+.\end{aligned}$$

We thus get

$$1\mathrm{st} + 3\mathrm{rd} = -\frac{[u+1]^*[u_1-u''-P+(c+1)/2]^*_+[u+P+1]^*_+[1]^{*3}}{[u]^*[u_1-u''+c/2]^*[P+1/2]^*_+[P-1/2]^*_+[u+1/2]^*}.$$

Therefore $\mathrm{Res}_{u'=u_2+c/2} G(u') \frac{dz'}{2\pi i z'} = 0$. The other cases can be treated in the similar way.

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